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FINITE ELEMENT APPLICATION TO TRANSIENT SCATTERING PROBLEMS.(U)  
MAR 79 A SANKAR, T C TONG

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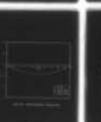
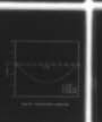
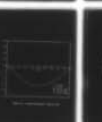
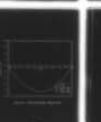
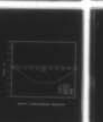
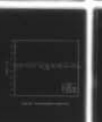
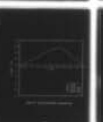
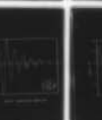
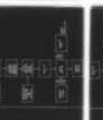
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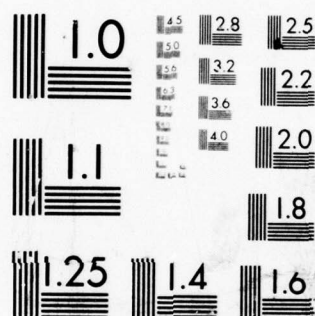
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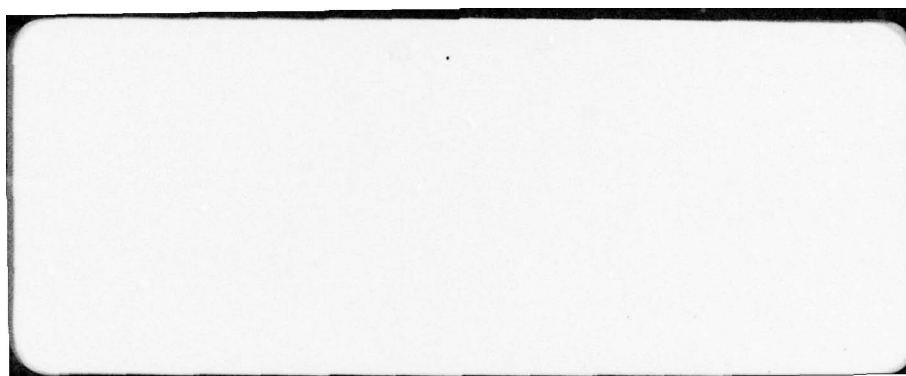
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FINITE ELEMENT APPLICATION  
TO  
TRANSIENT SCATTERING PROBLEMS

~~AFOSR~~ TECHNICAL REPORT ~~AFOSR~~ TRW-79-001

By  
A. Sankar  
T.C. Tong

Work Performed Under Air Force Contract  
No. F49620-78-C-0049 *June*

1 MARCH 1979

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Prepared for:

Air Force Office of Scientific Research  
Washington, D.C. 20332  
Air Force Office of Scientific Research  
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19 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER <b>AFOSR TR-79-0450</b>	2. GOVT ACCESSION NO. ---	3. RECIPIENT'S CATALOG NUMBER ---
4. TITLE (and Subtitle) <b>FINITE ELEMENT APPLICATION TO TRANSIENT SCATTERING PROBLEMS</b>	5. TYPE OF REPORT & PERIOD COVERED <b>Interim report</b> JAN 1978 - MAR 1979	
7. AUTHOR(s) <b>A./Sankar T.C./Tong</b>	8. CONTRACT OR GRANT NUMBER(s) <b>F49620-78-C-0049</b>	
9. PERFORMING ORGANIZATION NAME AND ADDRESS TRW DEFENSE AND SPACE SYSTEMS GROUP One Space Park, Redondo Beach, CA 90278	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS <b>61102F</b> <b>2301</b> <b>A3</b>	
11. CONTROLLING OFFICE NAME AND ADDRESS Director of Physics, AFOSR Attn: NP, Bldg 410, BOLLING AIR FORCE BASE Washington D.C. 20332	12. REPORT DATE <b>MAR 1979</b>	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) <b>73p</b>	13. NUMBER OF PAGES <b>67</b>	
15. SECURITY CLASS. (of this report) <b>UNCLASSIFIED</b>		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) <b>Approved for public release; distribution unlimited.</b>		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) ---		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Finite Element Method Electromagnetic Scattering Current Distribution on Thin Wire Time Domain Calculations		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Finite element method is applied to compute the induced current distributions on thin wires in the time domain. The required mathematical equations and computer code are developed. The predictions by this method compare very well with other techniques. Advantages of this approach are mentioned.		

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## 1. INTRODUCTION

### 1.0 OBJECTIVE

The primary objective of this study is to develop the Finite Element Method (FEM) for electromagnetic techniques to solve the problem of transient scattering directly in the time domain. The techniques will then be applied to compute the time-dependent currents induced on straight thin wires due to an incident plane wave Gaussian pulse.

### 1.1 Relevance of the Study

Transient electromagnetic response of structures such as strategic weapon systems and strategic command, communication and control systems to a nuclear electromagnetic pulse are of great concern from the point of view of their vulnerability and survival. Again, the importance of transient response cannot be overstated in radar target identification, electronic warfare and electronic countermeasures. For example, the impulse response can give a useful characterization of each radar target since such a response contains all necessary radar information in a compact and understandable form. Strategic systems should be designed to survive the nuclear transients. Thus, an understanding of the response becomes mandatory to impact on and improve the designs of systems. Since the systems, in general, are complicated structures, they can only be solved numerically; hence efficient and economical numerical techniques are required. The method developed in this study is expected to offer such a tool.

The approach is based on a unique technique for the computation of currents and fields on arbitrary structures excited by arbitrary sources. This technique called "Finite Element Method for Electromagnetics (FEMEM)" predicated on a variational principle governing the physics of the problem and an approximation procedure to carry out the variational expression integration.

## 1.2 A General Discussion of Transient Solutions

There are essentially two approaches for solving linear electromagnetic problems. One is an indirect approach in which the physics of the problem is abstracted either by a differential or by an integral equation with frequency as the variable. The equations are solved in the frequency domain and then the time domain solution is obtained by inverse Fourier transform. In the other approach, the governing equations are formulated and solved in the time domain.

In the time domain, the problem can be formulated either in terms of differential or integral equation. From a numerical solution point of view, the integral equation approach offers definite advantages over the differential equation approach with respect to solution stability and imposing boundary conditions.

## 1.3 The Finite Element Method in the Transient Domain

The finite element method has been successfully applied to a host of static or steady state problems, including eigenvalue problems throughout the many engineering disciplines. The extension of the method to transient problems may be credited to Wilson and Nickell<sup>[1]</sup> in their study of the heat conduction equation. Most of the early papers in this area concerned solutions to the diffusion equation in one form or another. Although the wave equation has been considered generally by Oden<sup>[2]</sup> there appears to be no specific solutions to this equation for transient problems. In general, three different approaches are used in solving the time domain problem in conjunction with the FEM. They are:

(1) In this method, the transient solution is obtained by developing a recurrence relation with the ordinary finite element equations for the problem and then time-stepping progressively. This technique will be further discussed later.

(2) This method depends on the idea of incorporating the time dimension directly into the finite element analysis as another one of the unknown nodal degrees of freedom of the systems. In this manner, time is discretized, as well as the spatial variables. Here the time span of



interest is divided into finite elements. Thus, the initial value problem is converted to a boundary value problem. Solutions for all intervals of time are obtained simultaneously, with nodes on each wire or surface  $t = \text{constant}$  defining the configuration of the system at that time. The increase in problem size due to the added time dimension is a disadvantage.

(3) In this approach, the solution is obtained by the mode superposition method. This technique is also known as the normal mode method or as modal analysis. The basis of this method is that the modal matrix of the eigenvalue problem can be used to diagonalize the problem and thus decouple the multiple degrees of freedom problem to give several one-degrees of freedom problem.

One advantage of the mode superposition method over the direct integration methods is that it reduces the number of equations to be solved. Since the lower normal modes play a more significant role in the response than the higher modes, only the lower modes need to be used. This method has the disadvantage of requiring the eigenvalue problem solution. Again, if the number of degrees of freedom is large, the eigenvalue problem is difficult. Superposition method is applicable only to linear problems. Thus, it transpires that the modal superposition method is less general than the other two methods mentioned earlier.

However, it must be mentioned that the advantages inherent in the finite element formulations can be profitably used in all three methods. This report primarily concerns itself with the first method. The subsequent sections discuss the problem formulation, FEM methodology, code development, numerical solutions, discussions, conclusions and recommendations.

## 2. MATHEMATICAL FORMULATION

### 2.0 FORMULATION OF THE VARIATIONAL INTEGRAL

The application of the FEM technique requires that we select the proper variational principle for the posed problem, express the functional involved in terms of approximate assumed current distribution functions which satisfy the boundary conditions and minimize this functional to obtain a set of governing equations which is then solved for the unknown current distributions at the nodes.

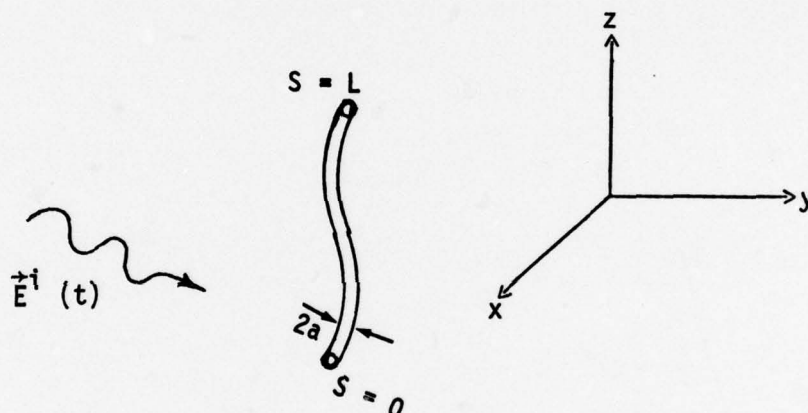


Figure 2-1. Geometry of the Problem

The wire is illuminated by a plane electromagnetic Gaussian pulse  $\vec{E}^i(t)$  with arbitrary polarization and angle of incidence.

Here the relations will be developed for general validity. Then, these will be specialized to the problem at hand. Let  $\vec{J}(\vec{r}, t)$  be the induced current on the perfectly conducting structure. The boundary condition applicable at the surface of a perfectly conducting body is given by

$$\hat{n} \times \vec{E}^t = \hat{n} \times (\vec{E}^s + \vec{E}^i) = 0 \quad (2-1)$$

where  $\hat{n}$  is unit normal to the surface and  $\vec{E}^t$ ,  $\vec{E}^i$  and  $\vec{E}^s$  are the total,



incident and scattered fields, respectively. This implies that the tangential electric field is zero.

The variational form functional  $L(\vec{J})$  governing the physics of the problem and containing the quantity of interest  $\vec{J}$  is given by

$$L(\vec{J}) = \oint_S \oint_{S'} \vec{J}(\vec{r}, t) \cdot \vec{K}(\vec{r}, \vec{r}', t) \cdot \vec{J}(\vec{r}', t) dr dr' - 2 \int_S \vec{J}(\vec{r}, t) \cdot \vec{E}^i(\vec{r}, t) dr \quad (2-2)$$

where  $\oint_S$  and  $\oint_{S'}$  denote Cauchy principal value integrations over the structure and  $dr$  and  $dr'$  differential elements. The Kernel  $\vec{K}(\vec{r}, \vec{r}', t)$  is a complicated integro-differential operator and is given by

$$\vec{K}(\vec{r}, \vec{r}', t) \equiv \frac{1}{4\pi} \left\{ \frac{\mu_0}{R} \frac{\partial}{\partial \tau} + \eta_0 \frac{\vec{R}}{R^2} \cdot \vec{\nabla}' + \frac{1}{\epsilon_0} \frac{\vec{R}}{R^3} \int_0^\tau dt \vec{\nabla}' \cdot \right\}, \quad \tau = t - \frac{R}{c} \quad (2-3)$$

where  $\mu_0$ ,  $\epsilon_0$  and  $\eta_0$  are free space permeability, permittivity and impedance, respectively;  $\vec{R} = \vec{r} - \vec{r}'$ , the vector distance between the observation point  $\vec{r}$  and the source point  $\vec{r}'$ ;  $\vec{\nabla}' \cdot$  denotes the divergence operation in the source coordinates;  $\tau = t - \frac{R}{c}$  is the retarded time. It is easily seen that the variation of  $L(\vec{J})$  with respect to  $\vec{J}$  leads to the time domain electric field integral equation.

$$\delta L(\vec{J}) = \vec{E}^i(\vec{r}, t) \cdot \vec{J}(\vec{r}, t) - \frac{1}{4\pi} \int_S \left\{ \frac{\mu_0}{R} \frac{\partial}{\partial \tau} + \eta_0 \frac{\vec{R}}{R^2} \cdot \vec{\nabla}' + \frac{1}{\epsilon_0} \frac{\vec{R}}{R^3} \int_0^\tau dt \vec{\nabla}' \cdot \right\} \vec{J}(\vec{r}', t) dr' = 0, \quad \tau = t - \frac{R}{c} \quad (2-4)$$

Now that the variational form is set for the problem, the remainder of the FEM technique is a procedure for rendering  $L(\vec{J})$  stationary by using an expression for  $\vec{J}$ .

## 2.1 Solution of the Variational Integral Equation by the Finite Element Method

The FEM is primarily a numerical procedure for solving complex problems. The method was originally used in the field of structural mechanics; but since its roots belong in mathematics as a class of approximation procedure, it can be applied to a wider variety of problems in other areas. In the FEM, the region of interest is divided into sub-domains or finite elements, with some functional representation of the solution being adopted over the elements so that the parameters of the representation become unknowns of the problems. Usually the element parameters are the nodal values and their derivatives at the nodes. Although the region of the problem is discretized into elements, the whole domain remains as a continuum because of the imposed restriction on the continuity across element interfaces. The mathematical procedure of solving (2-2) by the FEM is discussed in the following sections.

### 2.1.1 Segmentation in the Space and Time Coordinates

Examination of Eq (2-2) shows that the source current at  $\underline{r}$  delayed by a time  $|\underline{r} - \underline{r}'|/c$  is affecting the current at the observation point  $\underline{r}$ . Because of this retardation effect, Eq (2-2) can be solved as an initial-valued problem by using a time marching procedure. This phenomenon can best be visualized by considering the space-time diagram as shown in Figure 2-2.

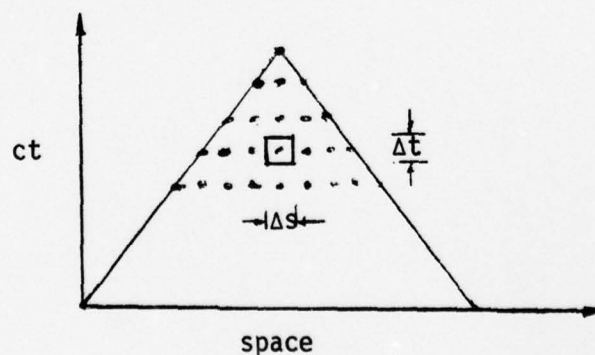


Figure 2-2. The Space-Time Diagram

In the space-time diagram each dot represents a space-time point; the solid lines are the characteristics of the wave equations and they separate the past and the future. To divide the current into the space and time coordinates, we expand the current in space and time as

$$\vec{J}(\vec{r}', t) = \sum_{i=1}^{N_s} \sum_{j=1}^{\infty} \vec{J}_{ij}(\vec{r}' - \vec{r}_i, t' - t_j) U(\vec{r}' - \vec{r}_i) U(t' - t_j) \quad (2-5)$$

$$\text{where } U(\vec{r}' - \vec{r}_i) = \begin{cases} 1 & \text{if } |\vec{r}' - \vec{r}_i| \leq \frac{\Delta_s}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2-6)$$

$$U(t' - t_j) = \begin{cases} 1 & \text{if } |t' - t_j| \leq \frac{\Delta_t}{2} \\ 0 & \text{otherwise} \end{cases}$$

with  $\Delta_s$  and  $\Delta_t$  as the spatial and temporal increments and  $J_{ij}$  represents the current value within the space segment  $i$  and time interval  $j$ . Therefore if one postulates that the incident field and all surface current on  $S$  are known or equal to zero for all time less than, say  $t_0$ , then the retarded time effect allows us to start the solution at time  $t_0$  and to view the integral equation as an initial-valued problem in a "marching on" procedure in time.

#### 2.1.2 The Subdivision of the Spatial Region (A Generalized Approach)

The region  $R$  is subdivided into discrete sub-regions or elements, each of the same general form, as shown in Figure 2-3, with the boundaries of each element being plane or curvilinear faces, and with the adjacent boundaries of any pair of elements being coincident. Commonly used elements for surfaces are triangular or polygonal form. At similar positions in each element, a number of points are identified as nodes. They are generally at the vertices of the elements, and at positions such as the center of an edge, the centroid of a face or the centroid of the element volume.

Let us denote the nodal values of the solution  $\phi$  at the  $p^{\text{th}}$  node as  $\phi_p$ . Let the number of elements into which region  $R$  is subdivided be  $N^t$ ,

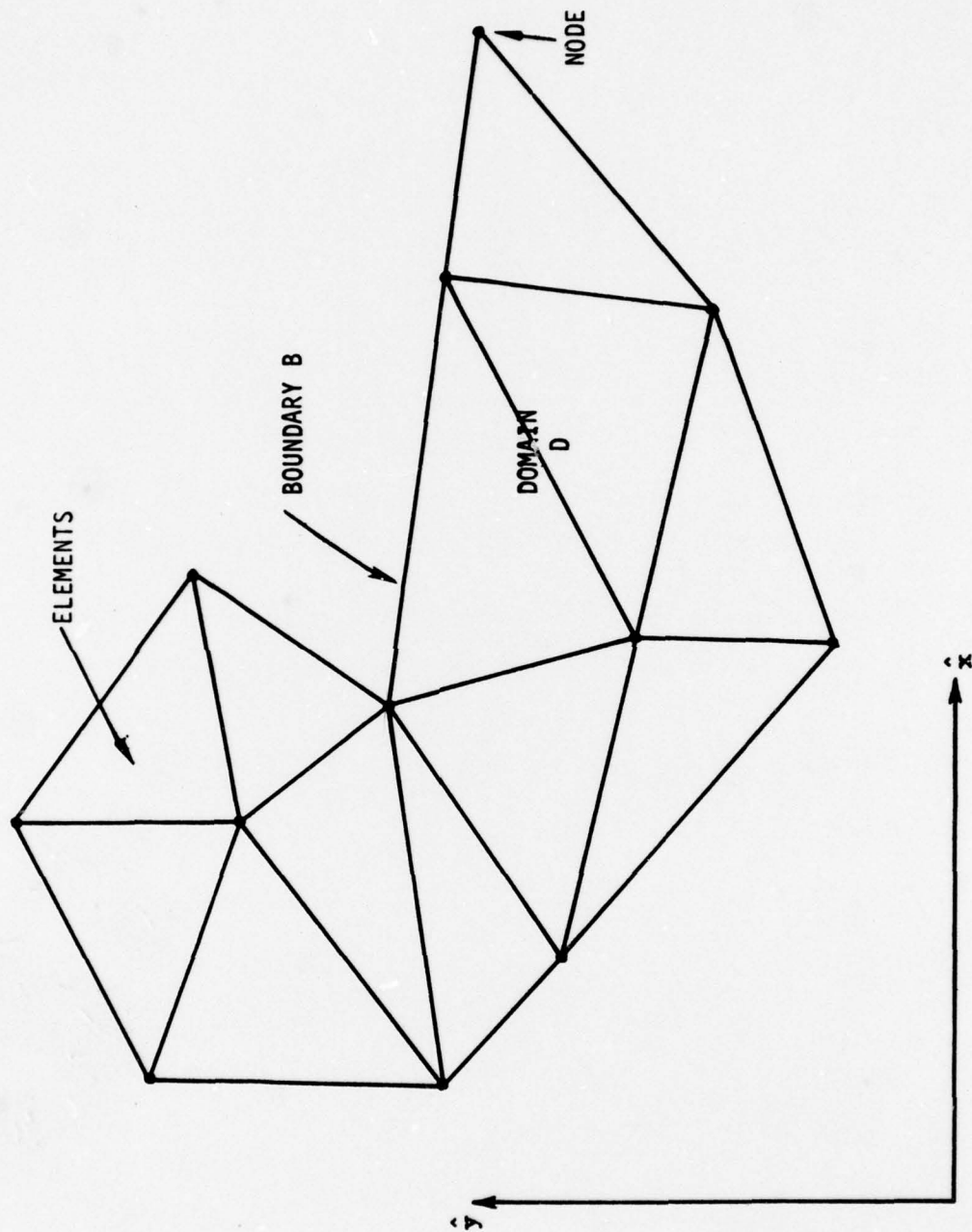


Figure 2-3. Subdivision of the Region into Finite Elements



and the total number of nodes in  $R = D + B$  (Boundary) be  $n_d$  and  $n_b$ . The Total number of nodes in a single element be  $n_s$ . Then the nodal values of  $\phi$  can be generally expressed as a column vector

$$\{\phi\} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \vdots \\ \phi_{n_d} \\ \vdots \\ \vdots \\ \phi_{n_d + n_b} \end{bmatrix} \quad (2-7)$$

### 2.1.3 The Element Shape Function

To solve Eq (2-2) by the FEM, one needs to define some shape functions or interpolation functions. These functions allow us to express the solution  $\phi$  at any position of  $R$  in terms of only the nodal values  $\{\phi\}$ . Therefore, we assume that the solution  $\phi$  can be described in functional forms, element by element, across the region, i.e., can be defined piecewise over the region. Within each element, it will be supposed that  $\phi$  can be described by a linear combination of functions  $N_1^e, N_2^e, \dots, N_k^e, \dots, N_s^e$ , and nodal values  $\phi_1^e, \phi_2^e, \dots, \phi_k^e, \dots, \phi_s^e$ , thus

$$\phi = \sum_e N_1^e \phi_1^e + N_2^e \phi_2^e + N_3^e \phi_3^e + \dots + N_k^e \phi_k^e + \dots + N_s^e \phi_s^e, \quad (2-8)$$

or, in matrix notation

$$\phi = \sum_e (N_1^e \ N_2^e \ \dots \ N_k^e \ \dots \ N_s^e) \{\underline{\phi}^e\} \quad (2-9)$$

$$= \sum_e (\underline{N}^e) \{\underline{\phi}^e\}. \quad (2-10)$$

Note that the superscript  $e$  is used here to identify a particular element.

The shape functions  $(\underline{N}^e)$  are restricted to being functions of positions. Since the true solution  $\phi$  is prescribed as being continuous and

with continuous derivatives (up to some order) across the region, the piecewise representation (2-10) should have the same properties. Therefore the shape functions are restricted by the following conditions:

1.  $N_j^e = 1$  at the  $j^{\text{th}}$  node
2.  $= 0$  outside element  $e$ , with the  $j^{\text{th}}$  node as one of its nodes
3.  $= N(\underline{r})$ , a position function within the elements.

In choosing the shape function, one has to pay attention to convergence in the FEM. Since it is recognized that the FEM solution to a problem with a given size of element is necessarily an approximation to the exact solution, there must be an assurance that successive finite element solutions using smaller and smaller elements will converge smoothly to the exact solution as the element size tends to a point. While comprehensive conditions ensuring convergence are not yet known for all types of linear problems, there are certain criteria that must be observed in order to obtain convergent solutions:

(1) Completeness

This means that the piecewise representation (2-10) within the element of the variable/derivative in a key integral must be capable of representing any continuous function as the element size decreases. Mathematically, the piecewise representation calls for a complete set of functions such as a polynomial function with infinite number of terms. However, in a FEM representation, only a finite number of terms is taken. But as pointed out by Melosh [5] and by Zienkiewicz [6], a monotonic convergence can still be obtained if the number of terms used in the representation allow the variable/derivative up to and including order  $t$  to take up any constant value within the element, where  $t$  being the highest-order derivative of the variable in the variational functional.

(2) Compatibility

This means that the representation of the variable/derivative in a key integral of (2-4) must tend to the same continuity as the exact solution, across the inter-element boundaries, as the size decreases to a point. If for a given variational functional, the highest-order derivative

involved is of order  $n$ , the derivatives of order up to and including  $(n-1)$  are known as the principal derivatives of that variable. Presuming that the exact solution of the dependent variables are continuous with continuous derivatives up to at least order  $n$ . One weak requirement that the compatibility criterion is satisfied is to require that the variable and their principal derivatives are continuous in the shape function representation. This means that the highest-order derivative in a key integral will have a representation that is at worst piecewise continuous, in which case the representation will tend to be continuous as the element reduces to a point. In general, completeness and compatibility are sufficient conditions for convergence in variational finite element methods. However, these conditions are very strong and can be relaxed [7]. In practice, the shape functions will not be an exact representation of the true solution, but an approximate one, and the solution obtained will be similarly approximate.

#### 2.1.4 The Subdivision of the Functional

Since (2-2) represents essentially a quadratic function, we can write it as

$$\phi = \int_D F(u_1, u_2, u_3 \dots, u_d) dD, \text{ where} \quad (2-11)$$

$$\begin{aligned} F(u_1, u_2, u_3 \dots, u_d) = & a_{11} u_1^2 + a_{12} u_1 u_2 + a_{13} u_1 u_3 \\ & + a_{21} u_2 u_1 + a_{22} u_2^2 + \dots + a_{dd} u_d^2, \end{aligned} \quad (2-12)$$

and  $D$  represents the domain of integration which can be a line, surface or volume, and  $u_1, u_2, u_3 \dots, u_d$  represent the solution  $\phi$  and its various derivatives,  $\phi_x, \phi_{xy}, \phi_y \dots$ . In matrix notation (2-11) becomes

$$\phi = \int_D \{u\}^T [A] \{u\} dD \quad (2-13)$$

where  $[A]$  is a  $d \times d$  matrix and  $\{u\}$  a  $d \times 1$  column vector, or

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & & a_{2d} \\ \vdots & & & \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{bmatrix}, \quad (2-14)$$



$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix} \quad (2-15)$$

and superscript T denotes the transpose of a matrix. In general, the matrix elements  $a_{ij}$  are functions of the position.

If  $\phi^e$  is the contribution of an element to the total integration in (2-13), then this equation can be written as

$$\phi = \sum_{e=1}^{\ell} \phi^e = \sum_{e=1}^{\ell} \int_{D_e} \{u\}^T [A] \{u\} dD_e \quad (2-16)$$

where  $D_e$  represents the domain of element  $e$ , let us now consider a typical term  $u_r$  in  $\{u\}$   $r = 0, 1, 2, \dots$ . By definition,  $u_r$  is a spatial derivative of  $\phi$ , that is  $u_r = \frac{\partial \phi}{\partial \xi^r} = D_r \phi$ , where  $\xi^r$  represents a spatial variable of concern.

From (2-10) we have

$$u = (N^e) \{\phi^e\} \text{ in element } e.$$

Thus within element  $e$

$$u_r = D_r \phi = (D_r N^e) \{\phi^e\} = (U_r^e) \{\phi^e\} \quad (2-17)$$

where  $(U_r^e)$  represents the row vector for the  $r$ -derivative of the shape function. So applying (2-17) for every element, we obtain

$$\{u\} = (U) \{\phi^e\} \quad (2-18)$$

where

$$[U] = \begin{pmatrix} U_1^e \\ U_2^e \\ \vdots \\ U_d^e \end{pmatrix} = \begin{bmatrix} D_1 N_1^e & D_1 N_2^e & \dots & D_1 N_s^e \\ D_2 N_1^e & D_2 N_2^e & \dots & D_2 N_s^e \\ \vdots & \vdots & & \vdots \\ D_d N_1^e & D_d N_2^e & \dots & D_d N_s^e \end{bmatrix}. \quad (2-19)$$



Substitution of (2-18) into (2-13) yields

$$\phi = \sum_{e=1}^{\ell} \int_{D_e} \{\phi^e\}^T [U]^T [A] [U] \{\phi^e\} dD_e \quad (2-20)$$

which shows that  $\phi$  is now a function of the  $n_d$  nodal values  $\phi_1, \phi_2, \dots, \phi_{n_d}$ .

#### 2.1.5 The Stationary Condition

In order to solve (2-20) we have to invoke the variational principle. The condition that  $\phi$  is stationary is given by

$$\frac{\partial \phi}{\partial \phi_1} = \frac{\partial \phi}{\partial \phi_2} = \frac{\partial \phi}{\partial \phi_3} = \dots = \frac{\partial \phi}{\partial \phi_{n_d}} = 0 \quad , \quad (2-21)$$

or

$$\frac{\partial \phi}{\partial \{\phi\}} = \left\{ \begin{array}{c} \partial \phi / \partial \phi_1 \\ \partial \phi / \partial \phi_2 \\ \vdots \\ \partial \phi / \partial \phi_{n_d} \end{array} \right\} = \{0\} \quad . \quad (2-22)$$

From (2-16) we get

$$\sum_{e=1}^{\ell} \frac{\partial \phi^e}{\partial \{\phi\}} = 0 \quad . \quad (2-23)$$

#### 2.1.6 The Element Matrix Equation

To get the element matrix equation we have to combine (2-23) and (2-20). Considering the term  $\frac{\partial \phi^e}{\partial \{\phi\}}$  for an element  $e$  in (2-20), we get

$$\frac{\partial \phi^e}{\partial \{\phi^e\}} = \int_{D_e} \frac{\partial}{\partial \{\phi^e\}} \left[ \{\phi^e\}^T [U]^T [A] [U] \{\phi^e\} \right] dD_e \quad (2-24)$$

Note that a term  $\frac{\partial \phi^e}{\partial \phi_p}$  will be zero unless  $p$  is one of the element nodes identified by  $1, 2, \dots, k, \dots, s$ . Also note that the node identifiers  $1, 2, 3, \dots, s$  are not the same as the system node numbers which are used to represent the total number of nodes in  $D$ . For example, if the triangular element  $e$  has its three vertices identified in the system node numbers as 7, 9, and 5, then we can let its node identifiers (now  $s = 3$ ) as  $1 \leftrightarrow 7, 2 \leftrightarrow 9$  and  $3 \leftrightarrow 5$ . Therefore, the only elements in the column vector that are non-zero are those that, in terms of element node identifiers, are

$$\frac{\partial \phi^e}{\partial \phi_1}, \frac{\partial \phi^e}{\partial \phi_2}, \dots, \frac{\partial \phi^e}{\partial \phi_s}.$$

So (2-22) reduces to

$$\frac{\partial \phi^e}{\partial \{\phi^e\}} = \begin{Bmatrix} \partial \phi / \partial \phi_1^e \\ \partial \phi / \partial \phi_2^e \\ \vdots \\ \partial \phi / \partial \phi_s^e \end{Bmatrix}. \quad (2-25)$$

Letting

$$[B] = [U]^T [A] [U], \quad (2-26)$$

and using

$$\frac{\partial}{\partial \{Y\}} \{Y\}^T [Q] \{Y\} = 2[Q] \{Y\}, \quad (2-27)$$

we obtain from (2-24)

$$\frac{\partial \phi^e}{\partial \{\phi^e\}} = \int_{D_e} 2[A^1] \{\phi^e\} dD_e \quad (2-28)$$

where

$[A^1]$  is a  $s \times s$  matrix.

Since  $\{\phi^e\}$  is constant with respect to the integration we can write (2-28) as

$$\frac{\partial \phi^e}{\partial \{\phi^e\}} = [A^1 e] \{\phi^e\} \quad (2-29)$$

where

$$[A^{1e}] = \int_{D_e} 2[A^{1e}] dD_e \quad . \quad (2-30)$$

If we substitute (2-19) into (2-24) and carry out the mathematics, we will obtain for the  $ij^{th}$  element of  $[B^e]$  as

$$\begin{aligned} b_{ij} = \int_{D_e} 2 \bigg[ & D_1 N_i^e (a_{11} D_1 N_j^e + a_{12} D_2 N_j^e + \dots + a_{1d} D_d N_j^e) \\ & + D_2 N_i^e (a_{21} D_1 N_j^e + a_{22} D_2 N_j^e + \dots + a_{2d} D_d N_j^e) \\ & + \dots + \\ & + D_d N_i^e (a_{d1} D_1 N_j^e + a_{d2} D_2 N_j^e + \dots + a_{dd} D_d N_j^e) \bigg] dD_e \quad . \end{aligned} \quad (2-31)$$

Note that in (2-31) the subscripts on the  $N^e$  are in terms of the node identifiers, not system node numbers.

Note that the shape functions are explicitly defined functions of spatial variables. The integrand of a particular term, say

$$\int_{D_e} 2(D_2 N_i^e) a_{21}(D_1 N_j^e) dD_e \quad ,$$

could be evaluated as an explicit function of  $x$ ,  $y$  and  $z$ . If  $a_{ij}$  are constant coefficients, the prescribed integration over the defined domain  $D_e$  of the element would, in consequence, evaluate the term as a scalar. The integration, if simple, could be carried out analytically. However, if  $a_{ij}$  are complex functions of  $x$ ,  $y$  and  $z$ , then the integration would generally require a numerical solution. Therefore, the computational time involved in a problem depends very much on whether  $a_{ij}$  are simple or complex functions.

### 2.1.7 The Boundary Condition

It is known in boundary-value problems that the solution is not unique unless it meets all the required boundary conditions. However, in the variational finite element methods, if the specified boundary conditions are natural boundary conditions for the problem, then it can be shown that the class of admissible functions is not required to satisfy these. In order to illustrate the treatment of the boundary condition in the matrix equation (2-29) let us assume a Dirichlet boundary condition such that

$$\phi = g(x,y,z) \text{ on } B. \quad (2-32)$$

Using (2-32), the  $n_b$  nodal values  $(\phi_p)_B$  for the boundary nodes on B can be calculated yielding  $n_b$  equations of the form

$$\begin{matrix} (0, 0, 0, \dots, 0, 1, 0, \dots, 0) \\ \uparrow \\ p^{\text{th}} \text{ Position} \end{matrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_p \\ \vdots \\ \phi_{n_b} \end{Bmatrix} = \begin{Bmatrix} g \\ g \\ g \\ \vdots \\ g \end{Bmatrix}, \quad (2-33)$$

which implies that if  $\phi_p$  satisfies the boundary condition and hence it is a constant value, then  $\frac{\partial \phi^e}{\partial \phi_p} = 0$  for an element containing node p. Thus to include the B.C. in the element matrix equation, the simplest procedure is to replace the  $p^{\text{th}}$  row of the matrix  $[A^{1e}]$  in (2-29) by the row matrix of (2-33). In other words, if p is a boundary-condition node, put zeros in the  $p^{\text{th}}$  row of the  $[A^{1e}]$  in (2-29) except for a 1 in the diagonal position and put in the  $p^{\text{th}}$  row of the driving vector the boundary value given by (2-32).



### 3. THIN WIRE SCATTERING

#### 3.0 APPLICATION OF THE FEM TO THE SCATTERING OF A GAUSSIAN PULSE BY A THIN CONDUCTING WIRE

The geometry of the problem is given in Figure 3-1. A perfectly conducting cylindrical wire of length  $\ell$  and radius  $a$  is located in free space as noted before.

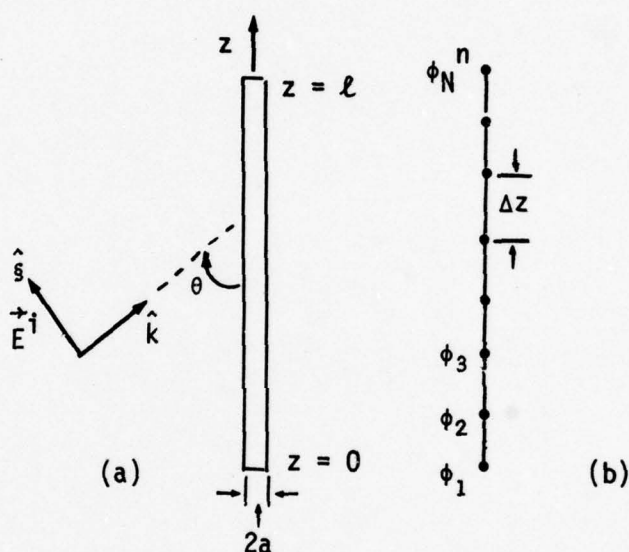


Figure 3-1. (a) Geometry of Thin-Wire Scatterer  
(b) Subdivision into Finite Elements

The wire is illuminated by a plane electromagnetic Gaussian pulse,

$\vec{E}^i(t) = e^{-[P(t-t_{\max})]^2} \hat{s}$ , having its  $E$  vector along  $\hat{s}$  and angle of incidence  $\theta$ , where  $\hat{s}$  is a unit vector perpendicular to  $\hat{k}$ , the propagation vector,  $P$  is the spread parameter and  $t_{\max}$ , the time at which the pulse reaches its maximum value. Since the wire is thin ( $\frac{a}{\lambda} \ll 1$ ), we can use the thin-wire approximation and assume that the current flows only in the  $z$  direction. Therefore, the variational equation (2-2) for the thin-wire is reduced to

$$F(\underline{z}, t) = \int_{\ell} \int_{\ell'} I_z(z, t) \cdot \underline{K}(z, z', \tau) \cdot I_z(z', \tau) dz' dz$$

$$- 2 \int_{\ell} I_z(z, t) \cdot E_z^i(z, t) dz \quad (3-1)$$

where

$$\underline{K}(z, z', \tau) \equiv \frac{1}{4\pi} \left\{ \frac{\mu_0}{R} \frac{\partial}{\partial \tau} + \frac{\eta_0 R}{R^2} \frac{\partial}{\partial z'} + \frac{1}{\epsilon_0} \frac{R}{R^3} \int_{-\infty}^{\tau} dt \frac{\partial}{\partial z'} \right\}, \quad \tau = t - \frac{R}{c}$$

$$\text{and } R = |\underline{z} - \underline{z}'| = \sqrt{(z - z')^2 + a^2} \quad (3-2)$$

To convert the integral equation we first divide the straight wire into  $N_s$  uniform segments with  $N^n$  number of nodes, and then express the current at any point lying inside a particular segment in terms of a shape function and the nodal values of that segment. Thus from (2-4) we have, after dropping the subscript  $z$

$$I^i(z', t') = \sum_{i=1}^{N_s} \sum_{j=1}^{\infty} I_{ij} (z' - z_i, t' - t_j) U(z' - z_i) U(t' - t_j) \quad (3-3)$$

$$\text{with } U(z' - z_i) = \begin{cases} 1 & \text{if } |z' - z_i| \leq \frac{\Delta z}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$U(t' - t_j) = \begin{cases} 1 & \text{if } |t' - t_j| \leq \frac{\Delta t}{2} \\ 0 & \text{otherwise} \end{cases},$$

where  $I_{ij}$  represents the shape function at the  $i^{\text{th}}$  segment and the  $j^{\text{th}}$  time interval. For a given time interval, say  $|t' - t_k| \leq \frac{\Delta t}{2}$ ,

$I_{ij}(z' - z_i, t' - t_j)$  is a function of space only. Therefore we can express the shape function  $I_{ij}$  in terms of spatial coordinates.

### 3.1 Spatial Shape Function

For the one-dimensional problem, we can represent the shape function by a polynomial of  $z$ . Thus, for  $|t' - t_j| \leq \frac{\Delta t}{2}$

$$I_{ij}(z' - z_i, t' - t_j) = N_1 + N_2 z + N_3 z^2 + \dots + N_k z^{k-1}. \quad (3-4)$$

The coefficients  $N_k$ 's are to be determined by the continuity requirement across element boundaries. Since the variational equation (2-2) involves first spatial derivative, it is necessary to use a polynomial of at least second order in order to meet the completeness and compatibility requirements for convergence. Thus we let

$$I_{ij}(z' - z_i, t' - t_j) = N_1 + N_2 z + N_3 z^2. \quad (3-5)$$

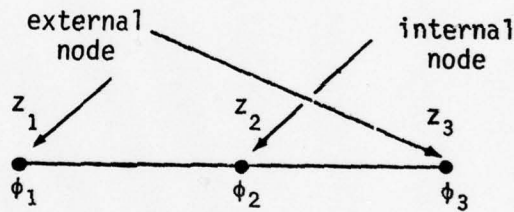


Figure 3-2. An Element with an Internal Node.

In order to determine  $N_1$ ,  $N_2$  and  $N_3$  uniquely, we have to pick an internal node in an element as shown in Figure 3-2, and require that

$$\begin{aligned} \phi_1 &= N_1 + N_2 z_1 + N_3 z_1^2 \\ \phi_2 &= N_1 + N_2 z_2 + N_3 z_2^2 \\ \phi_3 &= N_1 + N_2 z_3 + N_3 z_3^2, \end{aligned} \quad (3-6)$$

or

$$\phi_i = N_1 + N_2 z_i + N_3 z_i^2, \quad i = 1, 2, 3$$

where  $\phi_i$  is the nodal value of current at the  $i$ th node.

We can write (3-6) in matrix notation as

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{pmatrix}^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad (3-7)$$

After some matrix algebra manipulation we get

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \frac{1}{|Z|} \begin{pmatrix} z_2 z_3 (z_3 - z_2) & z_1 z_3 (z_1 - z_3) & z_1 z_2 (z_2 - z_1) \\ (z_2 - z_3)(z_2 + z_3) & (z_3 - z_1)(z_3 + z_1) & (z_2 - z_3)(z_2 + z_3) \\ (z_3 - z_2) & (z_1 - z_3) & (z_1 - z_2) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad (3-8)$$

$$\text{where } |Z| = (z_3 - z_2)(z_3 - z_1)(z_2 - z_1) \quad (3-9)$$

From (3-5) to (3-9) we obtain for an element connecting the  $i^{\text{th}}$  and the  $(i+1)^{\text{th}}$  nodes

$$\begin{aligned} I_{ij}(z' - z_i, t' - t_j) &= \frac{(z_m - z')(z_{i+1} - z')}{(z_m - z_i)(z_{i+1} - z_i)} \phi_i + \frac{(z_i - z')(z_{i+1} - z')}{(z_i - z_m)(z_{i+1} - z_m)} \phi_m \\ &\quad + \frac{(z_i - z')(z_m - z')}{(z_i - z_{i+1})(z_m - z_{i+1})} \phi_{i+1} \end{aligned} \quad (3-10)$$

$$z_i \leq z' \leq z_{i+1} \quad ,$$

where the subscript  $m$  denotes the internal node. Although the internal node can be placed at any position within the particular element, it is usually located at the midpoint of that element.

### 3.2 Time Derivative and Integration Interpolation

Since the kernel of the variational integral equation (2-2) contains also first time derivative, it is necessary to do temporal interpolation



over adjacent time intervals. A second-order Lagrangian interpolation is usually sufficient. Thus we let

$$\begin{aligned}
 I_{ij}(z'-z_i, t'-t_j) &= I_{ij}(z'-z_i) U(t'-t_j) \\
 &= I_{ij}(z'-z_i) \left[ \frac{(t'-t_j)(t'-t_{j+1})}{(t_j-t_{j-1})(t_{j+1}-t_{j-1})} \phi_{i,j-1} \right. \\
 &\quad + \frac{(t'-t_{j-1})(t'-t_{j+1})}{(t_{j-1}-t_j)(t_{j+1}-t_j)} \phi_{i,j} \\
 &\quad \left. + \frac{(t'-t_{j-1})(t'-t_j)}{(t_{j-1}-t_{j+1})(t_j-t_{j+1})} \phi_{i,j+1} \right]
 \end{aligned} \tag{3-11}$$

$$t_j - \frac{\Delta t}{2} \leq t' \leq t_j + \frac{\Delta t}{2},$$

where  $\phi_{ij}$  represents the nodal current at the  $i^{\text{th}}$  node at the  $j^{\text{th}}$  time interval.

To avoid extrapolation into the future, we have to interpolate the current at the  $j^{\text{th}}$  time step backwards to the  $j-1$  and  $j-2$  time steps when

$\frac{R}{c\Delta t} < 0.5$  such that

$$\begin{aligned}
 I_{ij}(z'-z_i, t'-t_j) &= I_{ij}(z'-z_i) \left[ \frac{(t'-t_{j-1})(t'-t_j)}{(t_{j-1}-t_{j-2})(t_j-t_{j-2})} \phi_{i,j-2} \right. \\
 &\quad + \frac{(t'-t_{j-2})(t'-t_j)}{(t_{j-2}-t_{j-1})(t_j-t_{j-1})} \phi_{i,j-1} + \frac{(t'-t_{j-2})(t'-t_{j-1})}{(t_{j-2}-t_j)(t_{j-1}-t_j)} \phi_{i,j} \left. \right].
 \end{aligned} \tag{3-12}$$

Equations (3-11) and (3-12) can be simply written as

$$I_{ij}(z'-z_i, t'-t_j) = I_{ij}(z'-z_i) \sum_{n=n_1}^{n_2} T_n \phi_{in} \tag{3-13}$$

where

$$\left. \begin{array}{l} n_1 = j-1 \\ n_2 = j+1 \end{array} \right\} \quad \text{if } \frac{R}{c\Delta t} \geq 0.5, \text{ and}$$

$$\left. \begin{array}{l} n_1 = j-2 \\ n_2 = j \end{array} \right\} \quad \text{if } \frac{R}{c\Delta t} < 0.5.$$

$T_n$  is either given by (3-11) or (3-12). From (3-11) and (3-12) we can also derive temporal derivative and integration as

$$\frac{\partial}{\partial t'} \left[ I_{ij} U(t'-t_j) \right] = I_{ij}(z'-z_i) \sum_{n=n_1}^{n_2} Q_n \phi_{in}, \quad (3-14)$$

and

$$\int_{\Delta t} I_{ij} U(t'-t_j) dt' = I_{ij}(z'-z_i) \sum_{n=n_1}^{n_2} D_n \phi_{in} \quad (3-15)$$

where  $Q_n$  and  $D_n$  can be obtained easily from (3-11) and (3-12).

### 3.3 Matrix Equation

Substitution of all the pertinent equations as derived above into (2-2) yields the  $v^{\text{th}}$  time step (i.e.,  $t = v\Delta t$ ).

$$\begin{aligned} F(\underline{j}) = & \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \int_{\Delta z_i} \int_{\Delta z_j} \sum_{\ell_i} N_{\ell_i} \phi_{\ell_i n} \left[ \frac{\mu_o}{R} \sum_n Q_n + \eta_o \frac{(z-z')}{R^2} \frac{\partial}{\partial z'} \right. \\ & + \frac{1}{\epsilon_o} \frac{(z-z')}{R^3} \int_0^T \sum_n D_n dt \left. \right] \left[ \sum_{\ell_j} N_{\ell_j} \phi_{\ell_j n} \right] dz' dz \\ & - 2 \sum_{i=1}^{N_s} \int_{\Delta z_i} \sum_{\ell_i} N_{\ell_i} \phi_{\ell_i} E_z^i(z_i, t_v) dz \end{aligned} \quad (3-16)$$

for  $z_i \leq z \leq z_{i+1}$ ,  $z_j \leq z' \leq z_{j+1}$

and  $|t_v - \frac{R}{c} - t_k| \leq \frac{\Delta t}{2}$  .

Note that  $i$  and  $j$  are used to denote the  $i^{\text{th}}$  and  $j^{\text{th}}$  elements while  $k$  is used to denote the  $k^{\text{th}}$  retarded time interval. The actual time interval is denoted by  $v$ . The summation  $\sum_{\ell_i}$  and  $\sum_n$  denote the summation process

over the spatial and temporal interpolations as given by (3-10) and (3-13).

To cast (3-16) into a matrix equation we invoke the stationary property of (3-16) by differentiating it with respect to each nodal current at the  $v^{\text{th}}$  time interval and setting the resulting equations equal to zero.

Thus

$$\begin{aligned} \frac{\partial F(J)}{\partial \phi_{pn}} = & \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \int_{\Delta_i} \int_{\Delta_j} \sum_{\ell_i} N_{\ell_i} \delta_{\ell_i p} \left[ \frac{\mu_0}{R} \sum_n Q_n + \eta_0 \frac{(z-z')}{R^2} \frac{\partial}{\partial z'} \right. \\ & + \left. \frac{1}{\epsilon_0} \frac{(z-z')}{R^3} \int_0^\tau \sum_n D_n dt \right] \left[ \sum_{\ell_j} N_{\ell_j} \phi_{\ell_j n} \right] dz' dz \\ & - \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \int_{\Delta_i} \int_{\Delta_j} \sum_{\ell_i} N_{\ell_i} \phi_{\ell_i n} \left[ \frac{\mu_0}{R} \sum_n Q_n + \eta_0 \frac{(z-z')}{R^2} \frac{\partial}{\partial z'} \right. \\ & + \left. \frac{1}{\epsilon_0} \frac{(z-z')}{R^3} \int_0^\tau \sum_n D_n dt \right] \sum_{\ell_j} N_{\ell_j} \delta_{\ell_j p} dz' dz \\ & - 2 \sum_{i=1}^{N_s} \int_{\Delta_i} \sum_{\ell_i} N_{\ell_i} E_z^i(z_i, t_v) \delta_{\ell_i p} dz = 0 \quad , \end{aligned} \quad (3-17)$$

where  $p = 1, 2, \dots, N^n$  and

$$\delta_{ip} = \begin{cases} 0 & \text{if } i \neq p \\ 1 & \text{if } i = p \end{cases} .$$

The final form of (3-17) can be symbolically written as

$$[S] (\phi_{iv}) = \left( E_z^i \mid_{t=t_v} \right) + (F) , \quad (3-18)$$

where  $[S]$  denotes the system matrix whose coefficients are functions of space and time. However its time dependence is the same for every time interval (assumed uniform). Therefore, matrix inversion is required only once.  $(F)$  denotes a known column vector containing information from previous computation.

The boundary condition imposed here is

$$\phi_{iv} = 0 \quad (3-19)$$

where  $i = 1$  and  $i = N^n$ .

### 3.4 Numerical Integration

By using the FEM, the integration over the entire wire is now reduced to a summation of integration over the individual elements. The integration in each element is carried out numerically by replacing the integration by its Riemann sum with unit-weighting coefficient. That is, if we divide the  $i^{\text{th}}$  element into  $N$  subdivisions, we have

$$\int_{\Delta_i} f(z) dz = \sum_{j=1}^N f(z_j) \Delta z \quad (3-20)$$

where  $\Delta_i$  = the domain of the  $i^{\text{th}}$  element

$\Delta z$  = the size of a subdivision

$z_j$  = the  $z$  coordinates of the center of the  $j^{\text{th}}$  subdivision of the  $i^{\text{th}}$  element.

### 3.5 Matrix Inversion

Since the problem is solved as an initial-value problem, it is not necessary to invert the matrix at each time step of solution. Matrix inversion is done only once at the first time step and the inverted matrix is stored to be used for the following-on time steps. Thus the solution after the first time step can be written as

$$(\phi_{iv}) = [S]^{-1} \left[ \left( E_z^i \mid_{t=t_v} \right) + (F) \right] \quad (3-21)$$



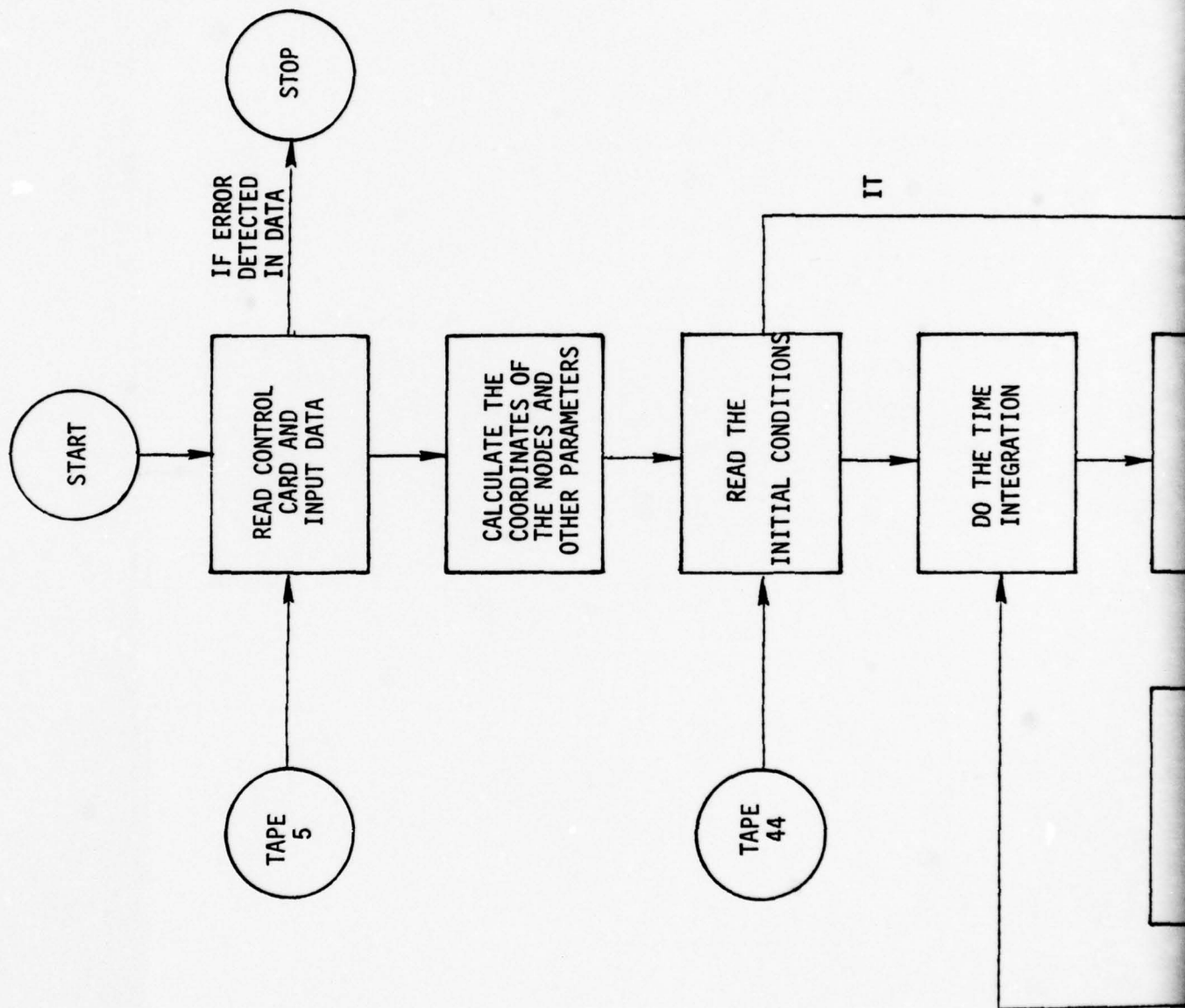
## 4. COMPUTER PROGRAM

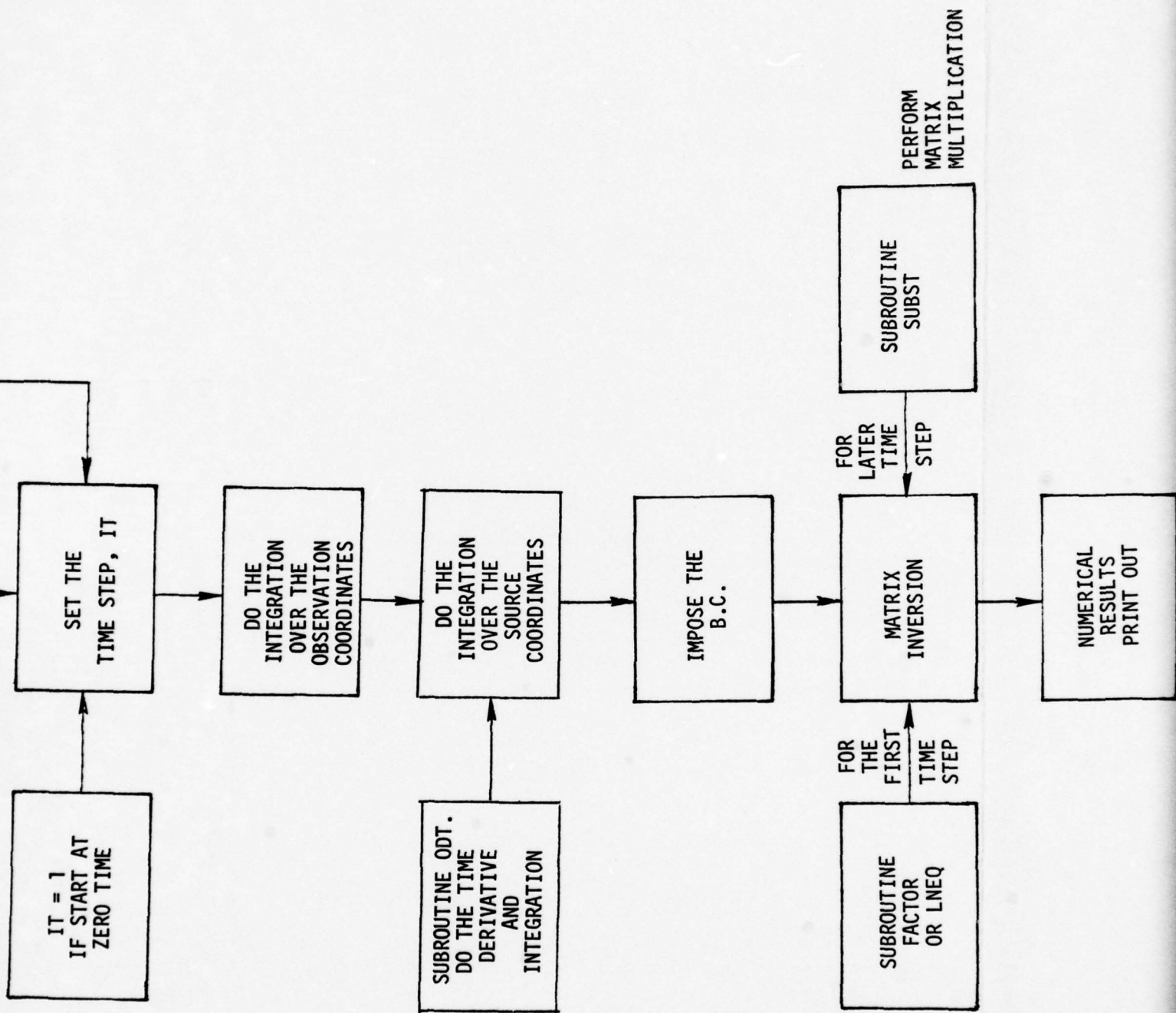
### 4.0 A BRIEF DESCRIPTION OF THE COMPUTER PROGRAM

The computer program developed for this study is called "TWFEM". It is written in the Fortran IV language. The program consists of a main program and four subroutines. The input to the programs are: the length of the wire, the radius of the wire, the number of elements into which the wire is divided, the size of the time step, the final time for the run, the angle of incidence, the gaussian pulse width, the time at which the gaussian incident field reaches its maximum, and a few control option parameters for running the program. The output of the program is the current distribution, the incident field strength and the segment excitation on each node at each time step. Most of these outputs are stored on a magnetic tape and can be saved for future use. Because of this, the program can use the results of the final time step in a previous run as the initial values for the new run. This capability is designed to save computational time by eliminating duplicated computation. The numerical integration is performed by a simple trapezoidal quadrature, and the matrix inversion is done only once using the gaussian elimination algorithm. To save computational time, many parameters are stored in common blocks.

### 4.1 Flow Chart

The structure of the computer program is given in a flow chart as shown in Figure 4-1.





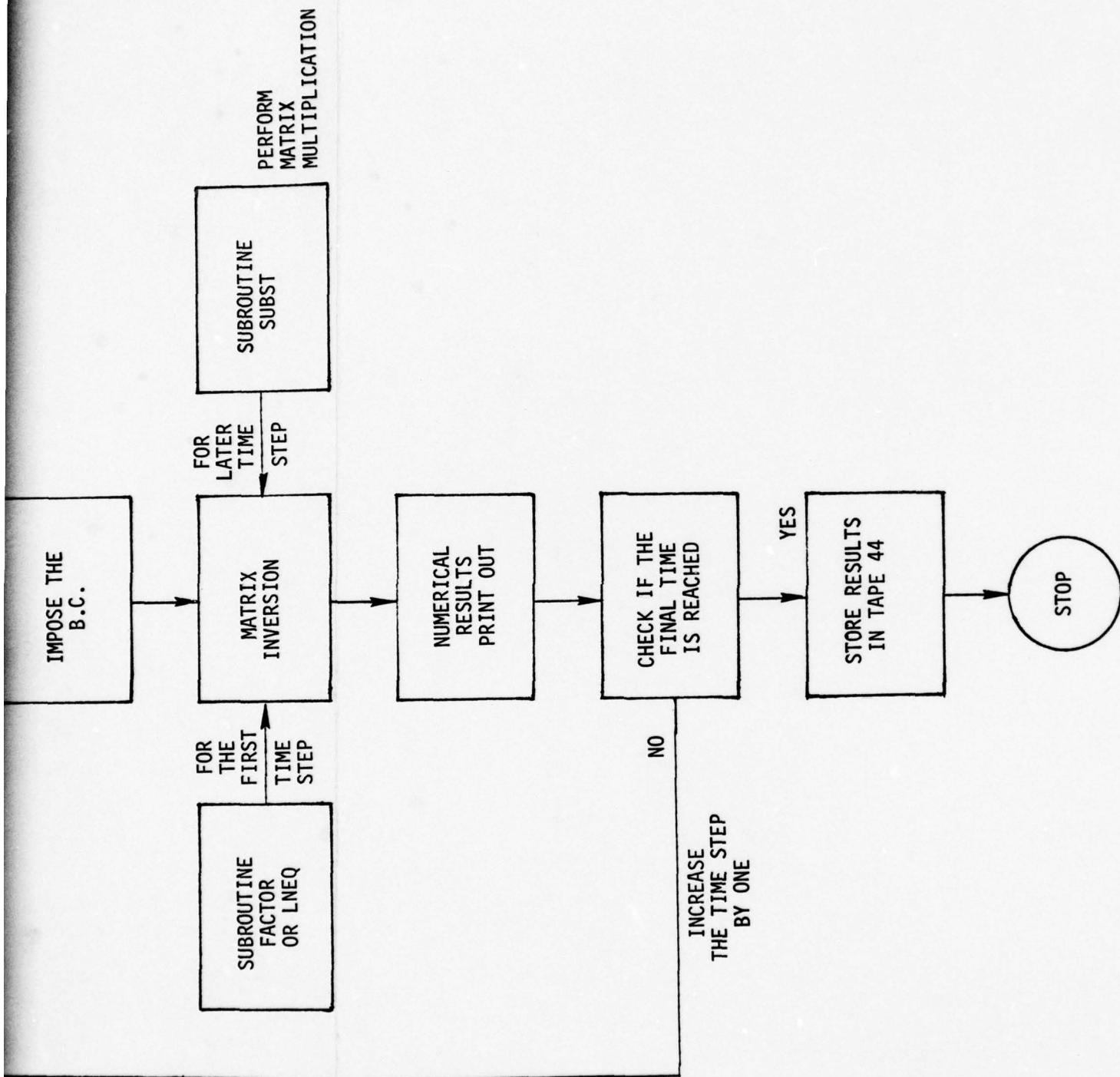


Figure 4-1. Flow Chart of Computer Program



## 5. RESULTS AND DISCUSSIONS

### 5.0 TRANSIENT CURRENTS

Figures 5-1 to 5-3 present the induced transient currents at the center of a thin wire when illuminated by a unit gaussian pulse ( $p = 9 \times 10^8 \text{ sec}^{-1}$ ,  $t_{\text{max}} = 1 \text{ nsec}$ ) at angles of incidence  $\theta = 30^\circ$ ,  $60^\circ$  and  $90^\circ$ . The length of the wire  $L$ , is .5 m and the shape factor  $\Omega = 2\ln\left(\frac{L}{a}\right) = 10.02$ . The wire is divided into five elements with eleven nodes (six external and five internal) and the time interval is taken to be 0.167 nsec. The current is seen to oscillate at the lowest characteristic frequency of the wire and decay in an exponential manner. The comparison between the FEM results and those obtained by using the method of moments code<sup>[8]</sup> are shown in Figure 5-3. It is seen that extremely good agreement is obtained between the current predictions by these two approaches. Figure 5-4 shows similar results for a wire of 1 m length. Again very good agreement is indicated between the FEM and the method of moments. By making the wire thinner, i.e., larger  $\Omega$ , the current oscillates at a slightly faster rate with smaller peaks. Figure 5-5 shows this behavior.

As is obvious from the above figures, the time-dependent induced currents on the wire depend on various factors including the incident field's magnitude, direction of arrival, polarization, pulse width, wire length and wire thickness. It is also interesting to note that the late time oscillation of the current is determined by the fundamental resonance frequency of the wire. It is also obvious that the current waveforms can be considered as the superposition of damped sinusoids.

#### 5.1 Current Distribution on the Entire Wire as a Function of Time

The current distribution over the entire length of the wire depends on the instant of observation. Figures 5-6 through 5-19 show the instantaneous current distributions along the wire at different time intervals ranging from  $t = 0.584 \text{ nsec}$  to  $t = 7.097 \text{ nsec}$ . The parameters used in

the computation are  $L = 0.5$  m,  $\Omega = 10.02$ ,  $\theta = 30^\circ$  and  $\Delta t = 0.167$  nsec. A careful examination of these plots shows the temporal development of the current along the wire. First, the current starts to build up from the end of the wire where the incident field pulse hits initially. As time goes on, the other part of the wire is also illuminated and the current pulse begins to travel with the velocity of propagation toward the other end. When the current pulse reaches the other end (Figure 5-8) the current pulse reverses its direction of propagation as the current cannot flow forward any farther. This phenomenon goes on and on until the current completely decays due to radiation loss.

Figures 5-20 through 5-33 show the snap shot type current distributions on the wire for normal incidence at fourteen different instants. Figure 5-23 shows the reverse polarity of the current distribution at  $t = 2.087$  nsec. As one should expect for normal incidence, the current distributions display the symmetry with respect to the center of the wire. As the time progresses (Figures 5-24 – 5-33) the current bounces back and forth with decreasing amplitude but maintaining the symmetry. Ultimately, the induced current will reradiate away.

## 5.2 Convergence and Computational Time

Figure 5-34 shows the convergence test for the current at the middle of a thin wire of length  $L \approx 0.5$  m and  $\Omega = 10.023$ . The number of elements used in the test are 3, 5 and 7 which correspond to 7, 11 and 15 nodes. It is seen that the numerical results show a better convergence in late time than in early time. This is understandable, since in an initial-value problem the initial values are sensitive to the step size used, and the step sizes used in this convergence test are chosen such that  $\frac{c\Delta t}{\Delta z} = 1$ , where  $\Delta t$  is the time step and  $\Delta z$  is the element size. The computational time for each run depends on the number of elements and the number of time steps used. A typical run in this study uses five elements (11 nodes) and 120 time steps and it takes about 100 sec on the CDC 6500 machine. However, it must be noted that the computer program has not been optimized to take into account various factors such as structure symmetry for the broadside illumination and possible analytical integration. Once this is done, the computational time can be reduced further.

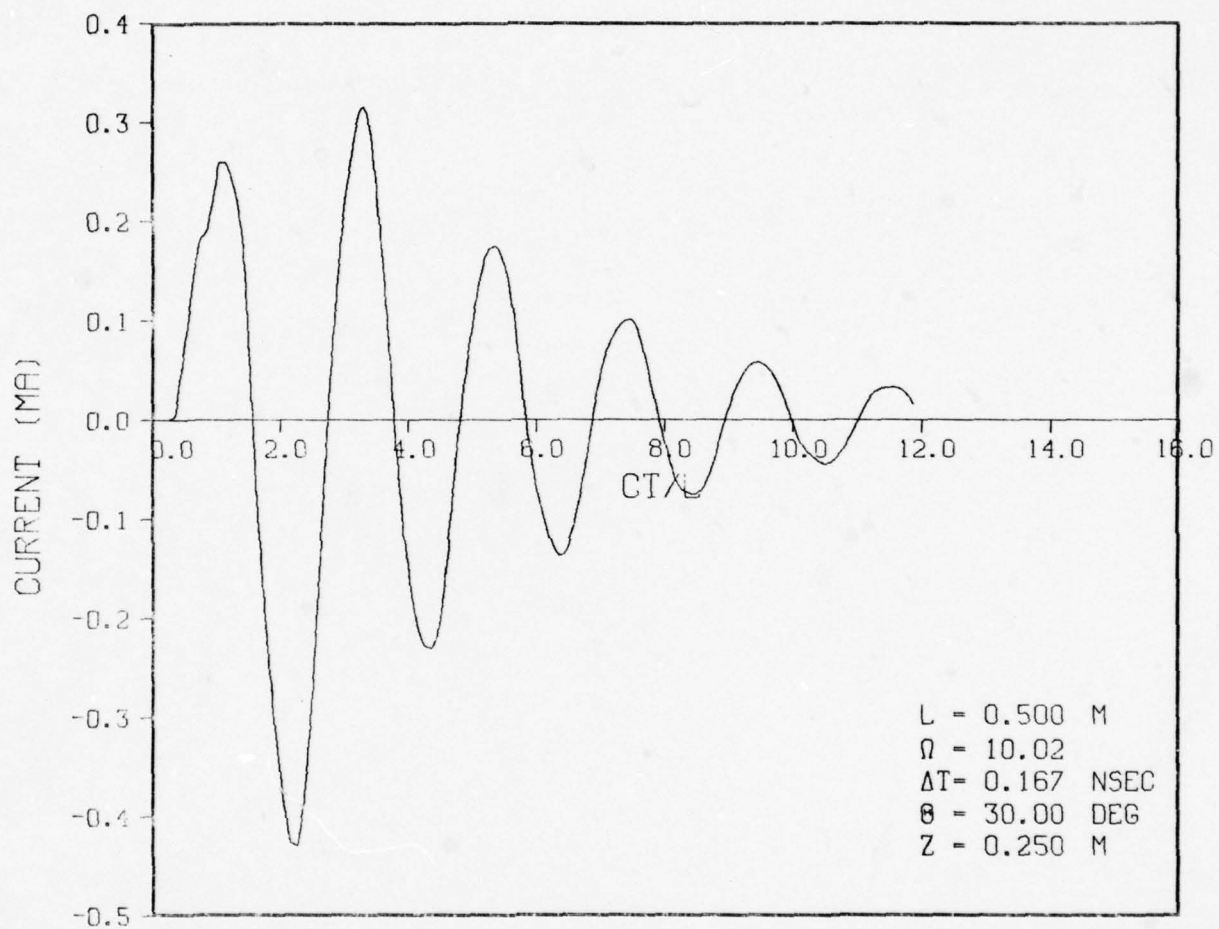


Figure 5-1. Transient Current — Gaussian Pulse

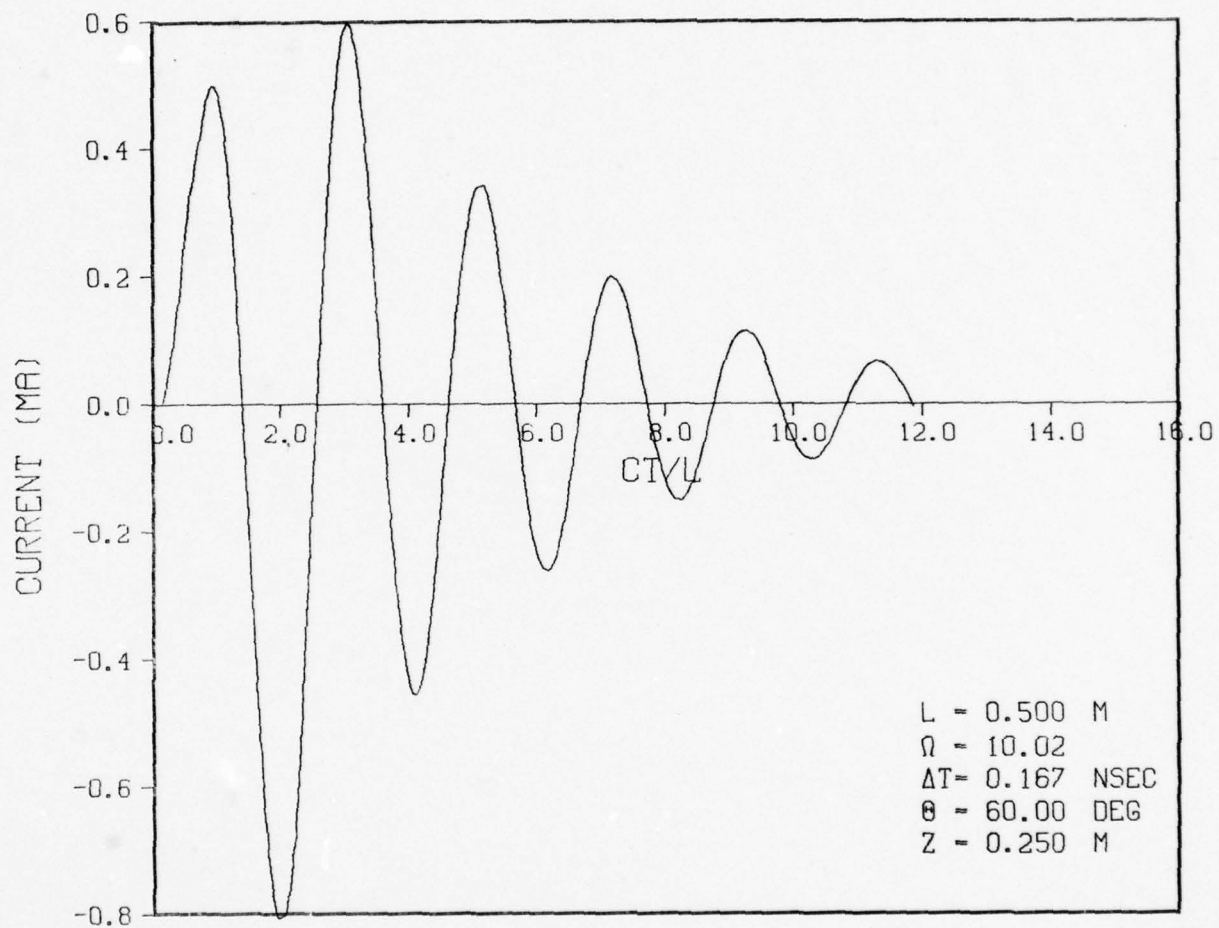


Figure 5-2. Transient Current — Gaussian Pulse



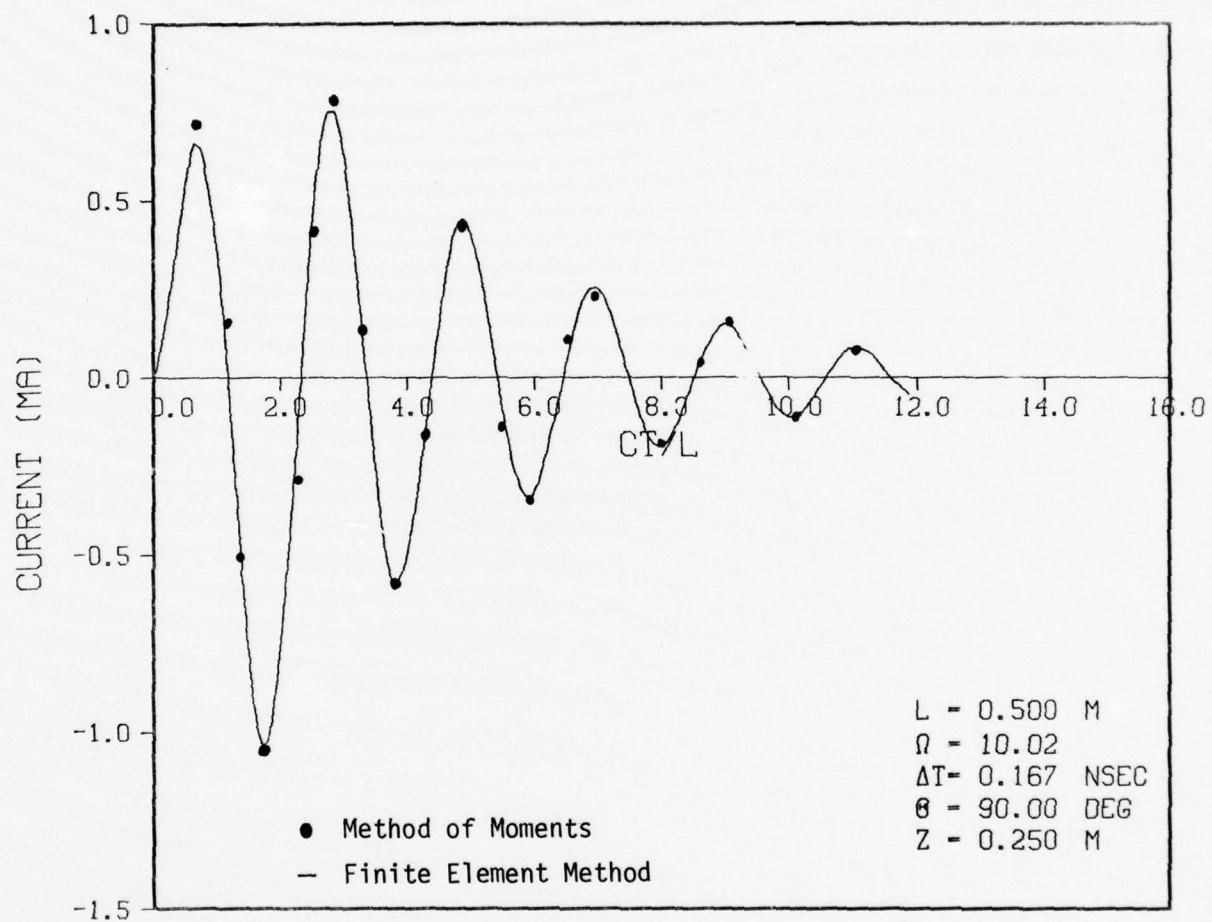


Figure 5-3. Transient Current — Gaussian Pulse

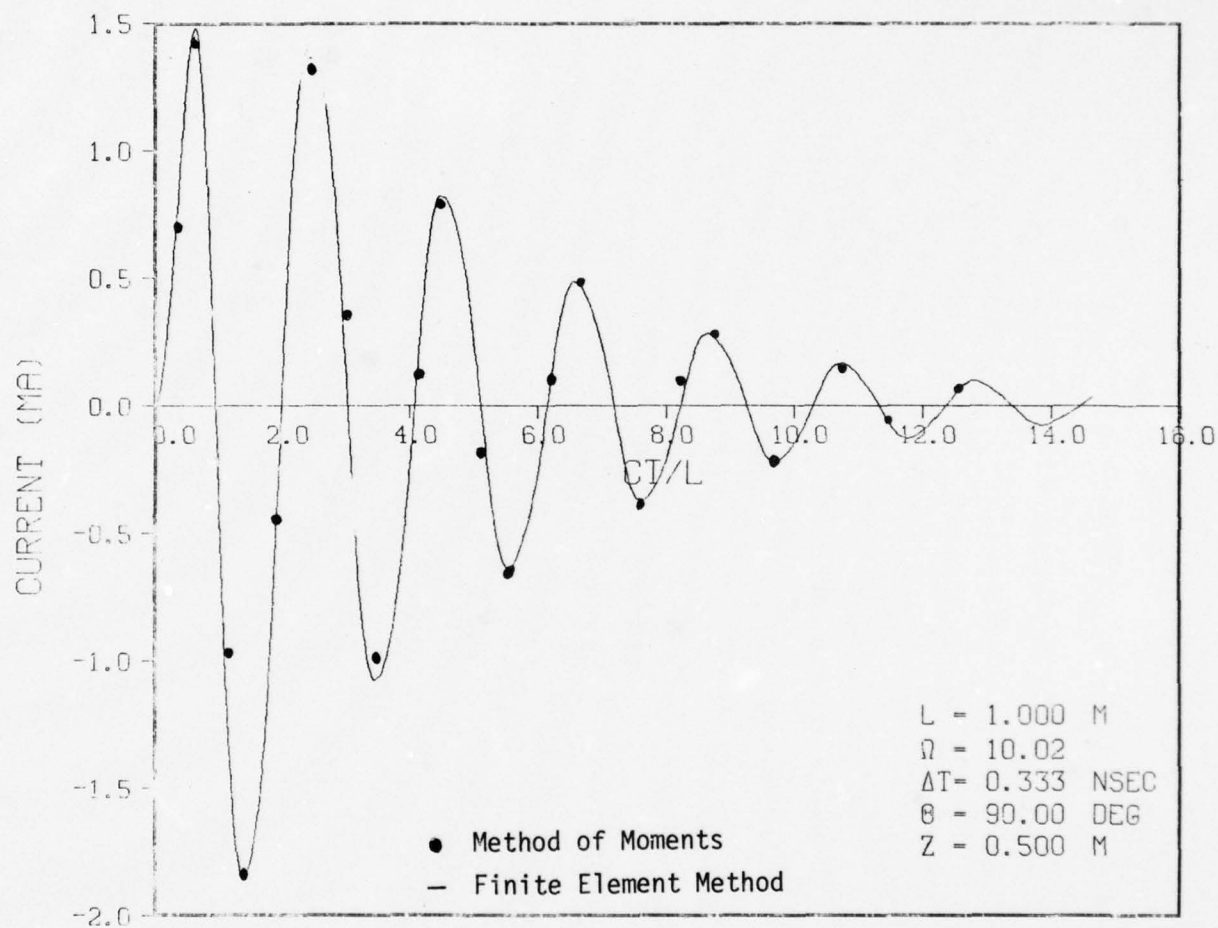


Figure 5-4. Transient Current — Gaussian Pulse

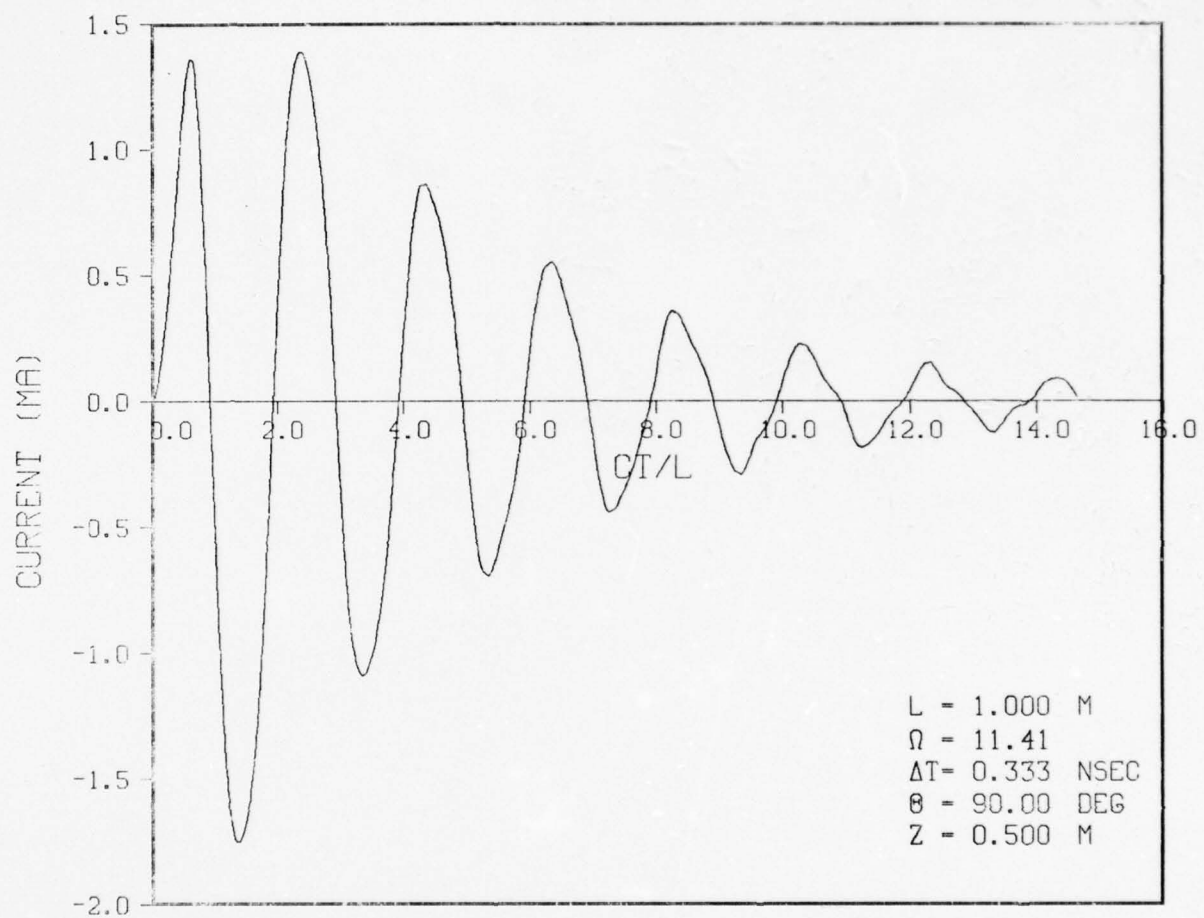


Figure 5-5. Transient Current — Gaussian Pulse

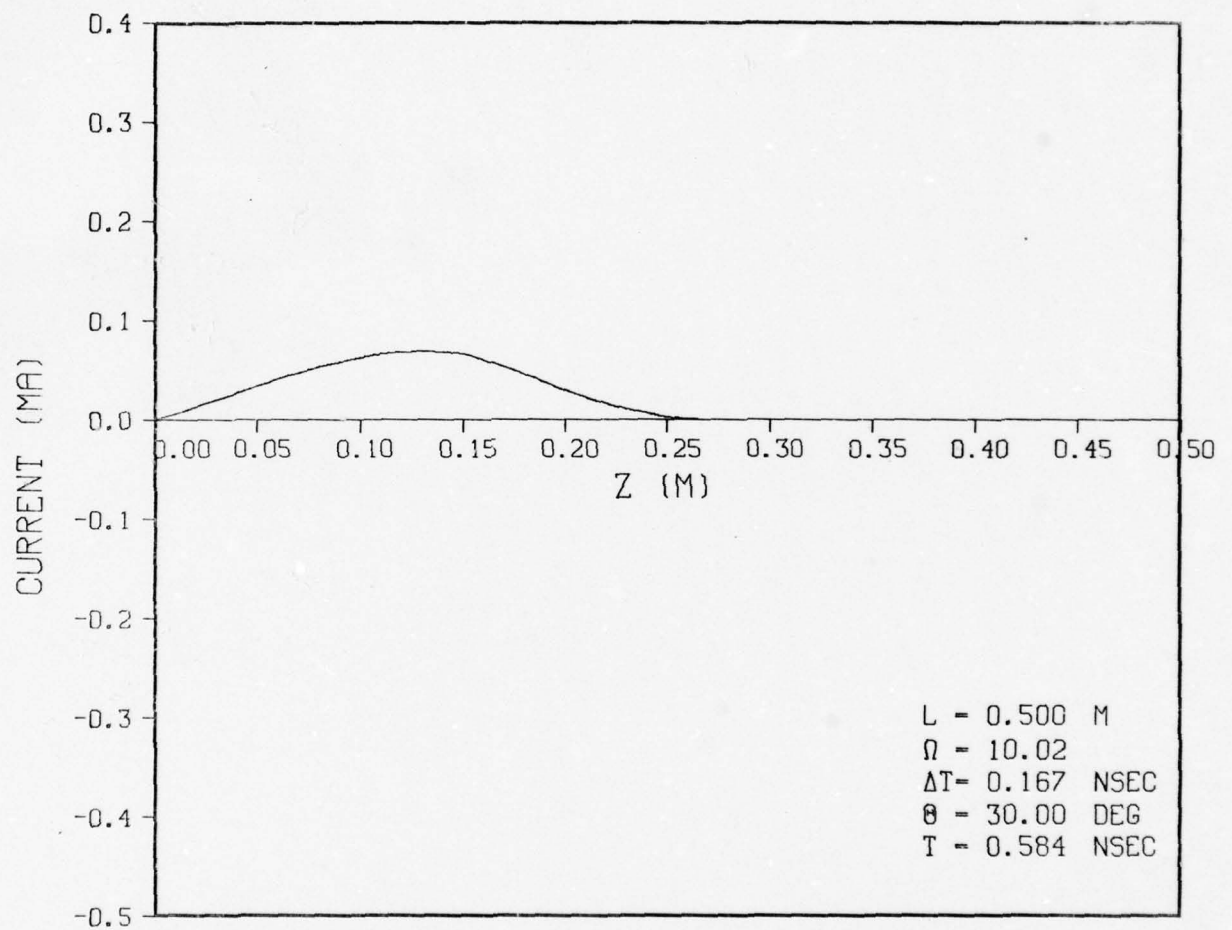


Figure 5-6. Current Distribution — Gaussian Pulse



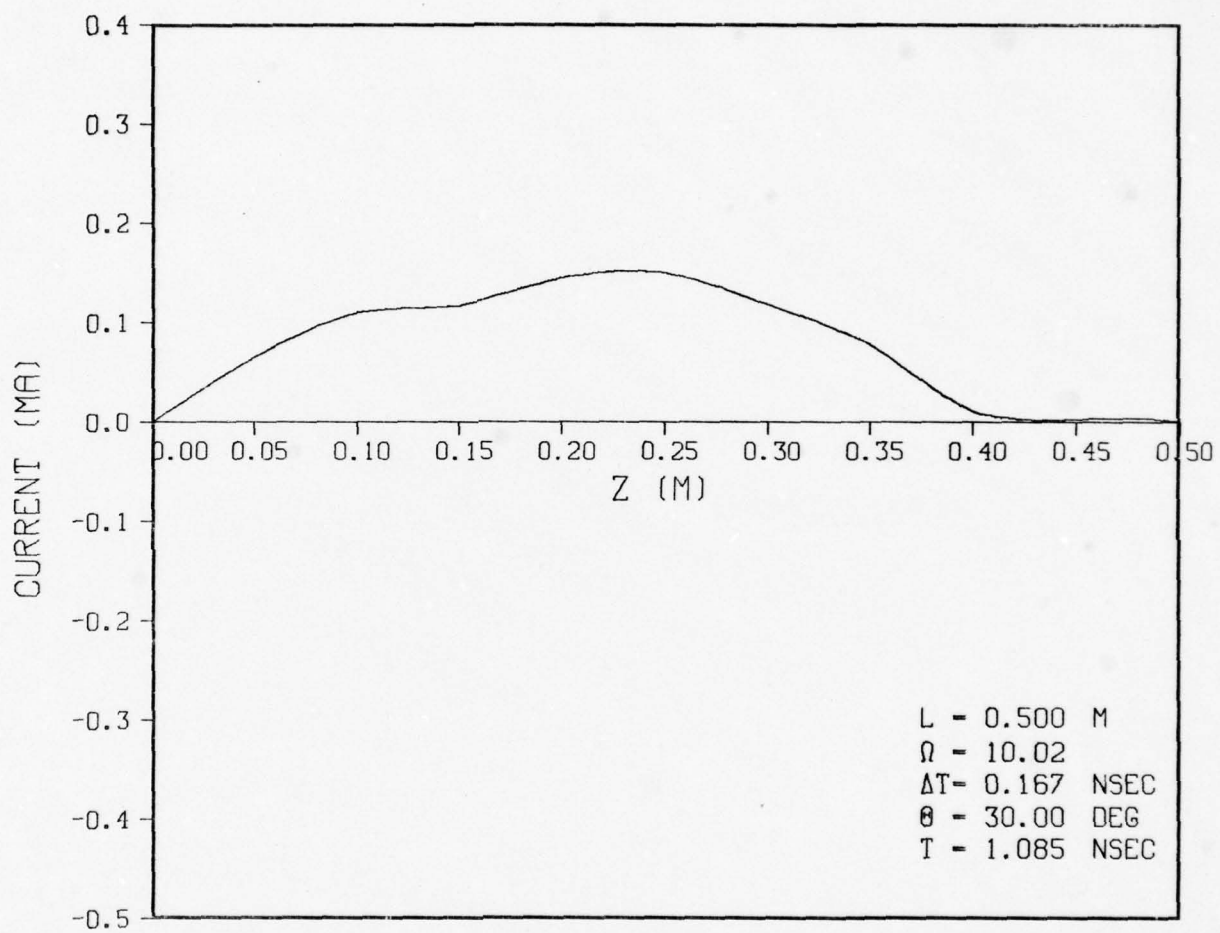


Figure 5-7. Current Distribution — Gaussian Pulse

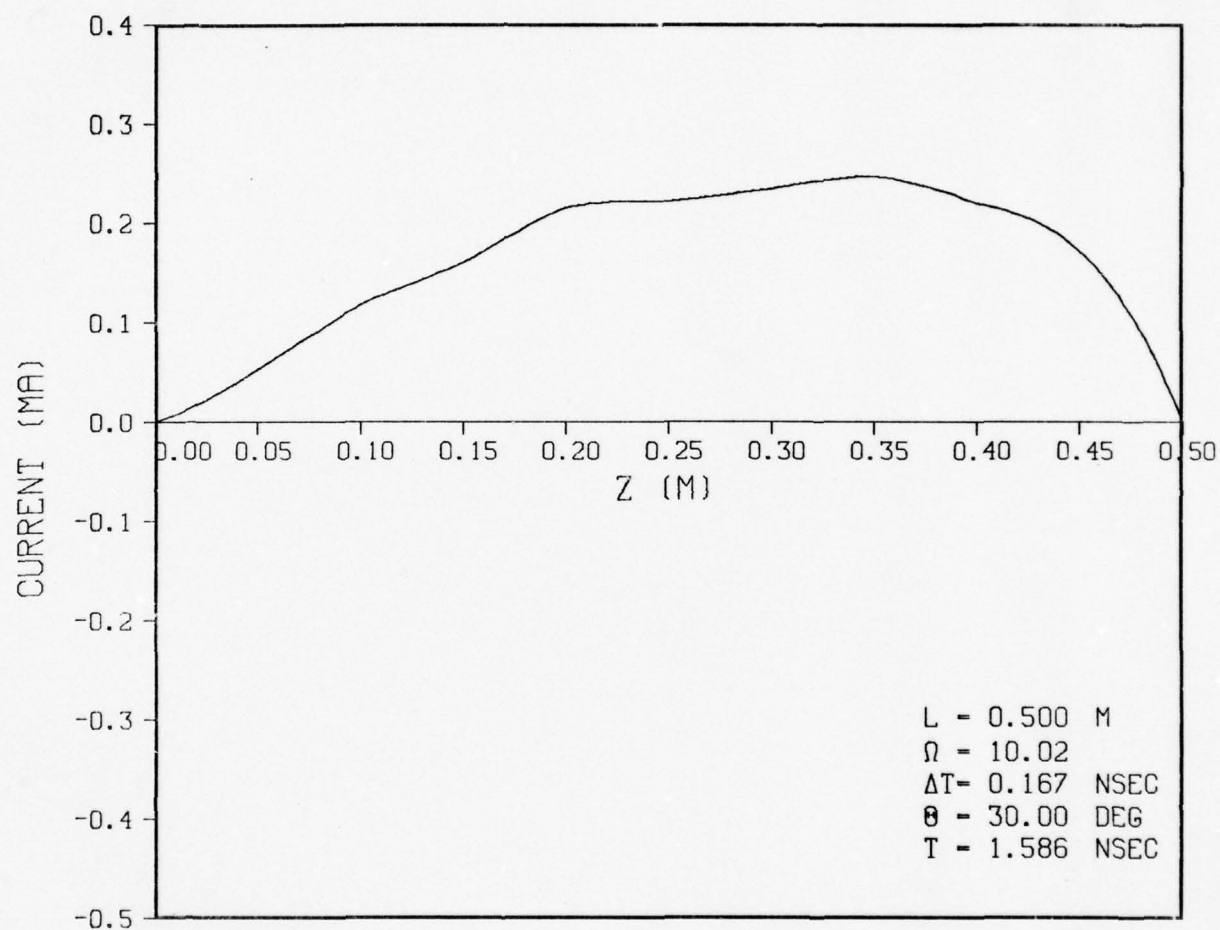


Figure 5-8. Current Distribution — Gaussian Pulse

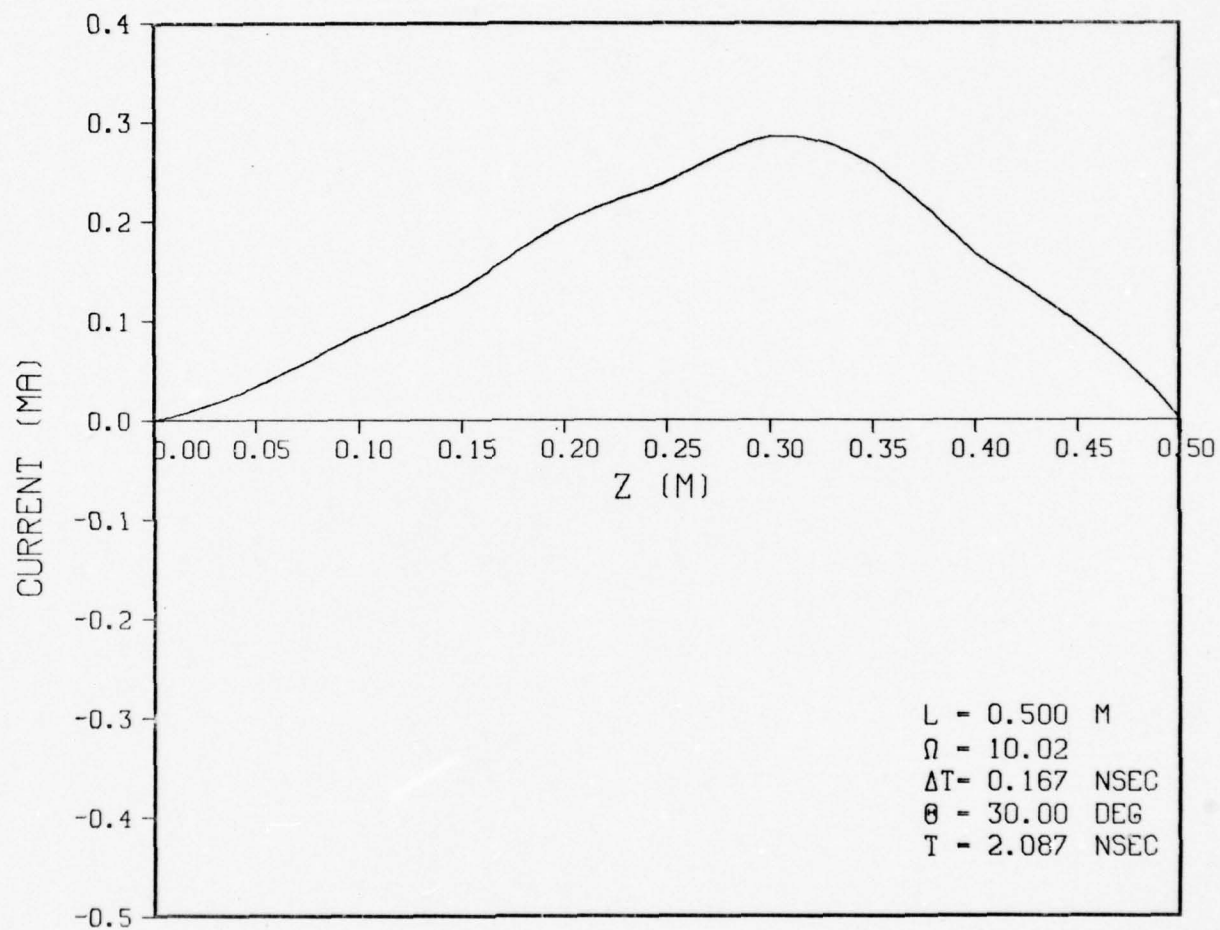


Figure 5-9. Current Distribution — Gaussian Pulse

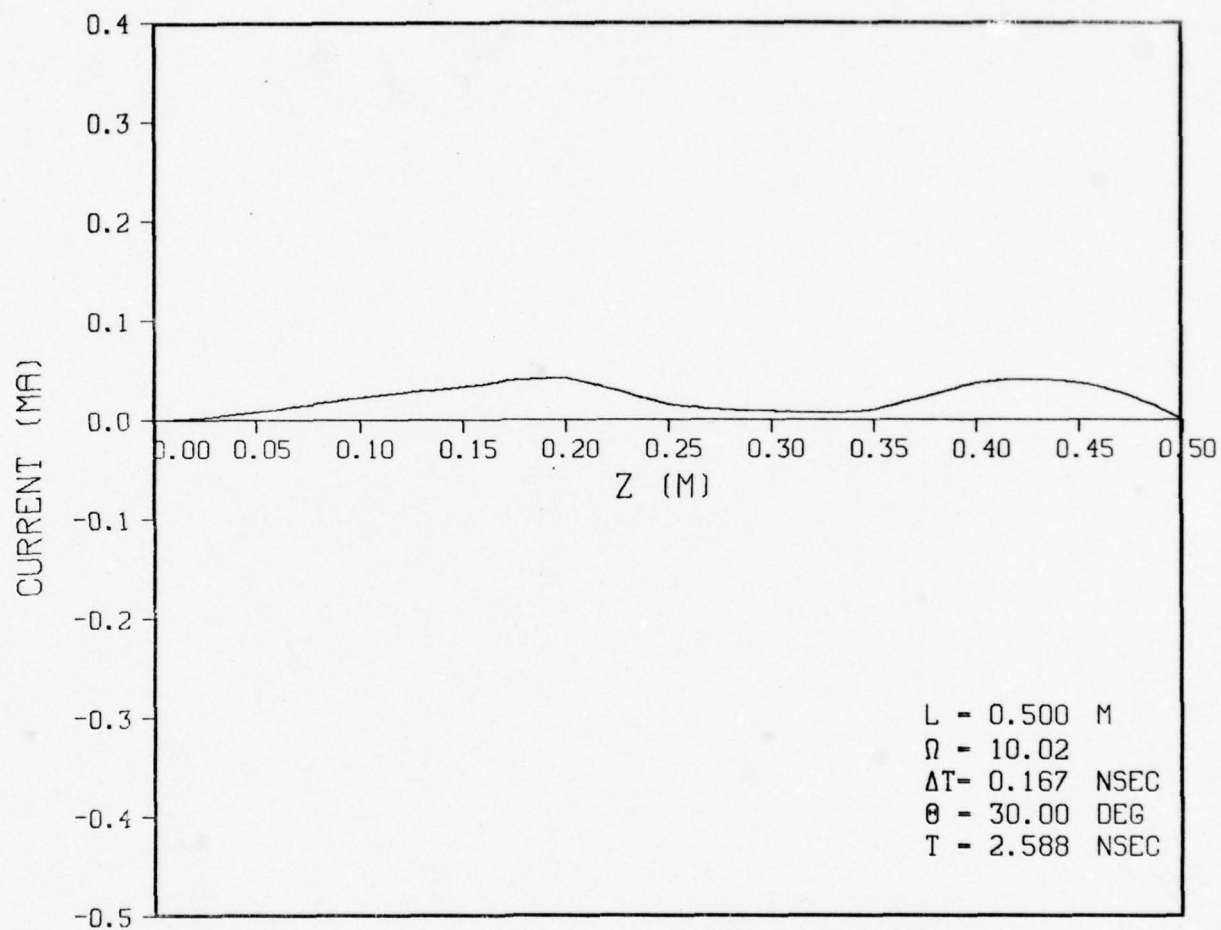


Figure 5-10. Current Distribution — Gaussian Pulse



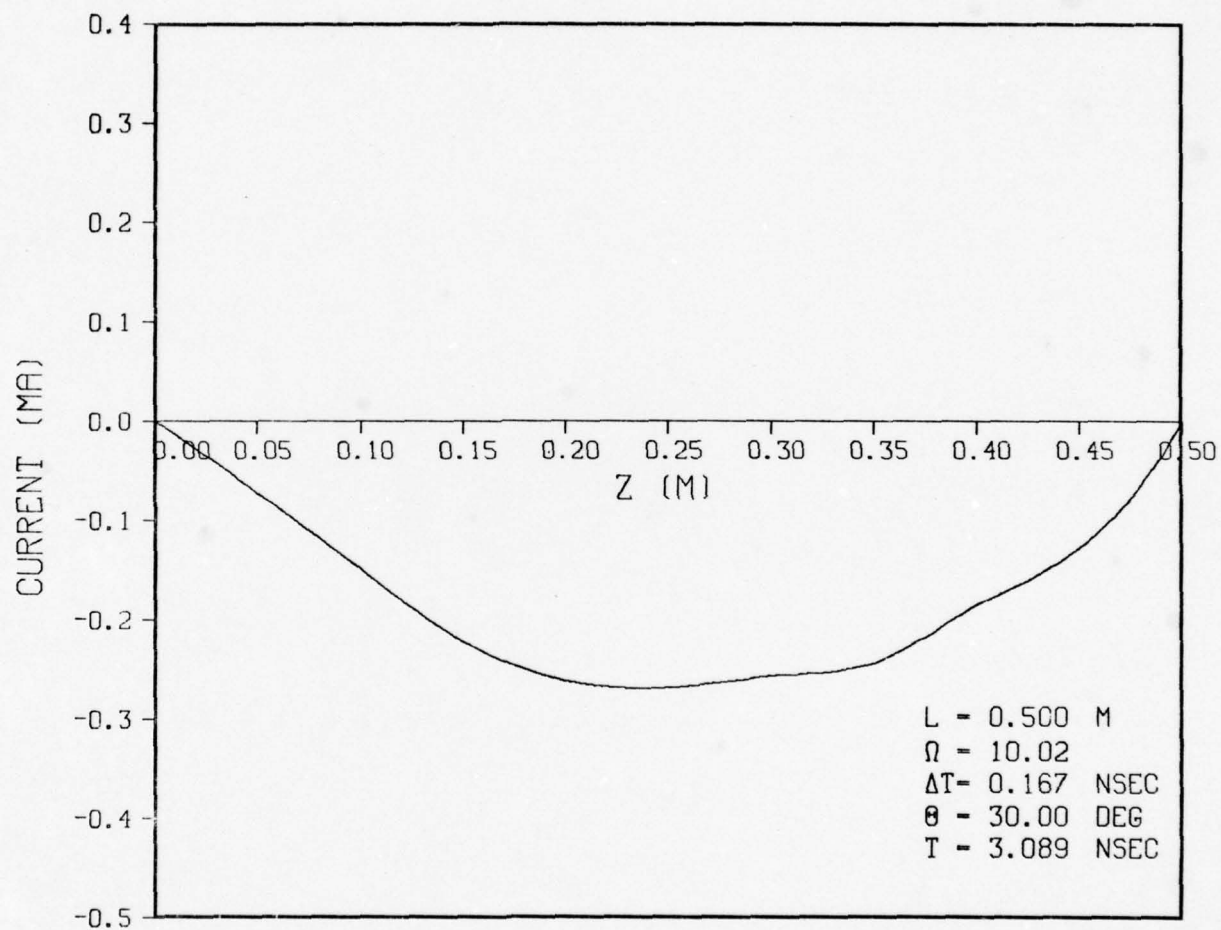


Figure 5-11. Current Distribution — Gaussian Pulse

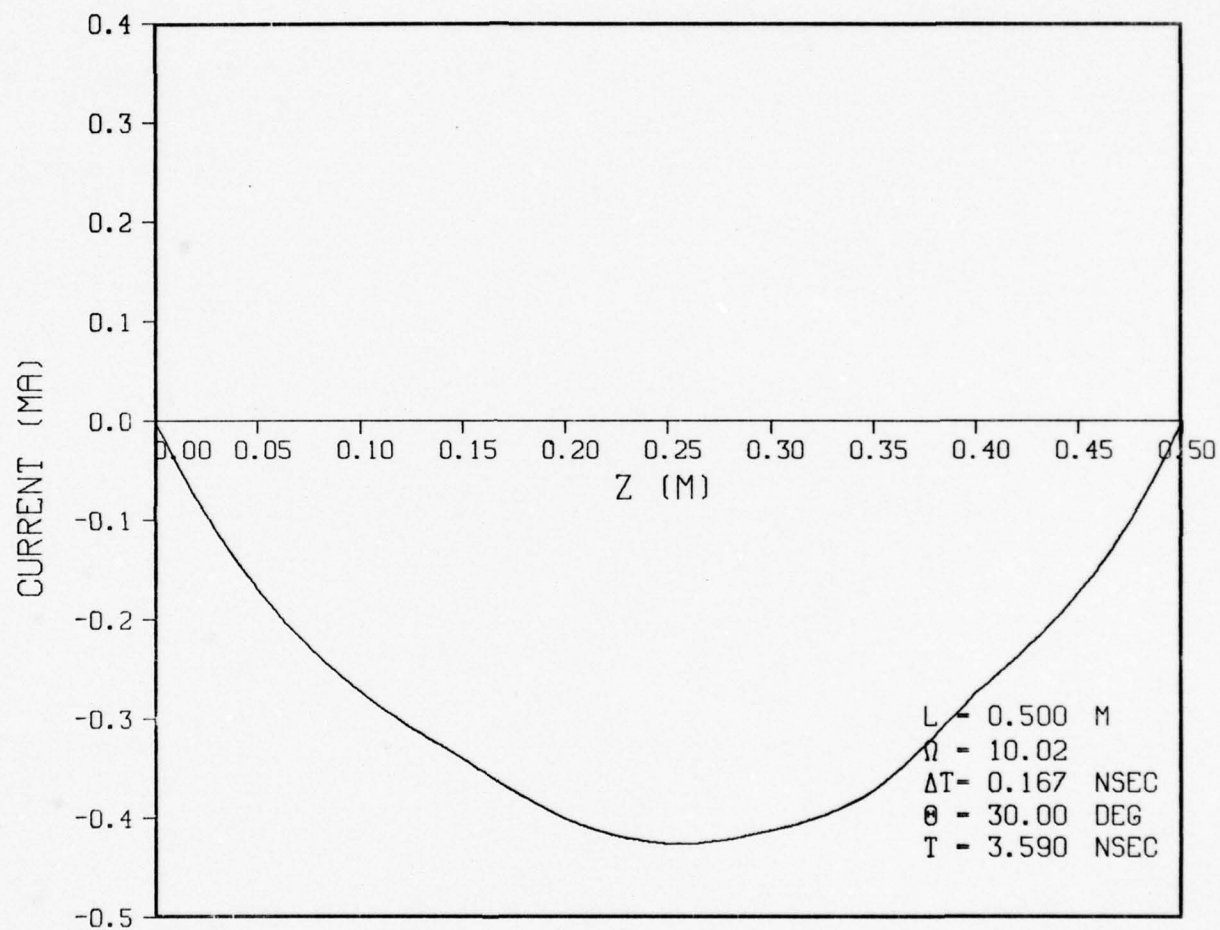


Figure 5-12. Current Distribution — Gaussian Pulse

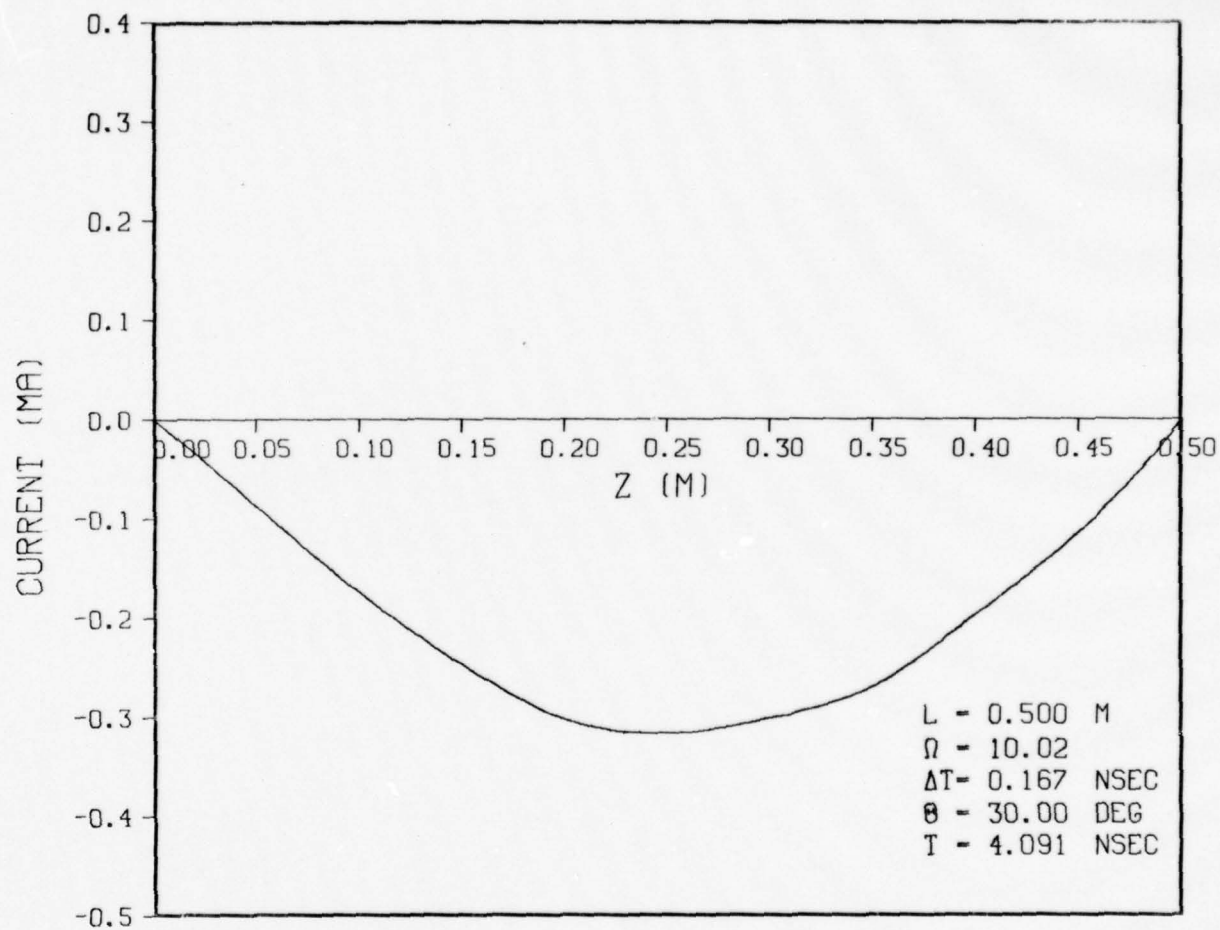


Figure 5-13. Current Distribution — Gaussian Pulse

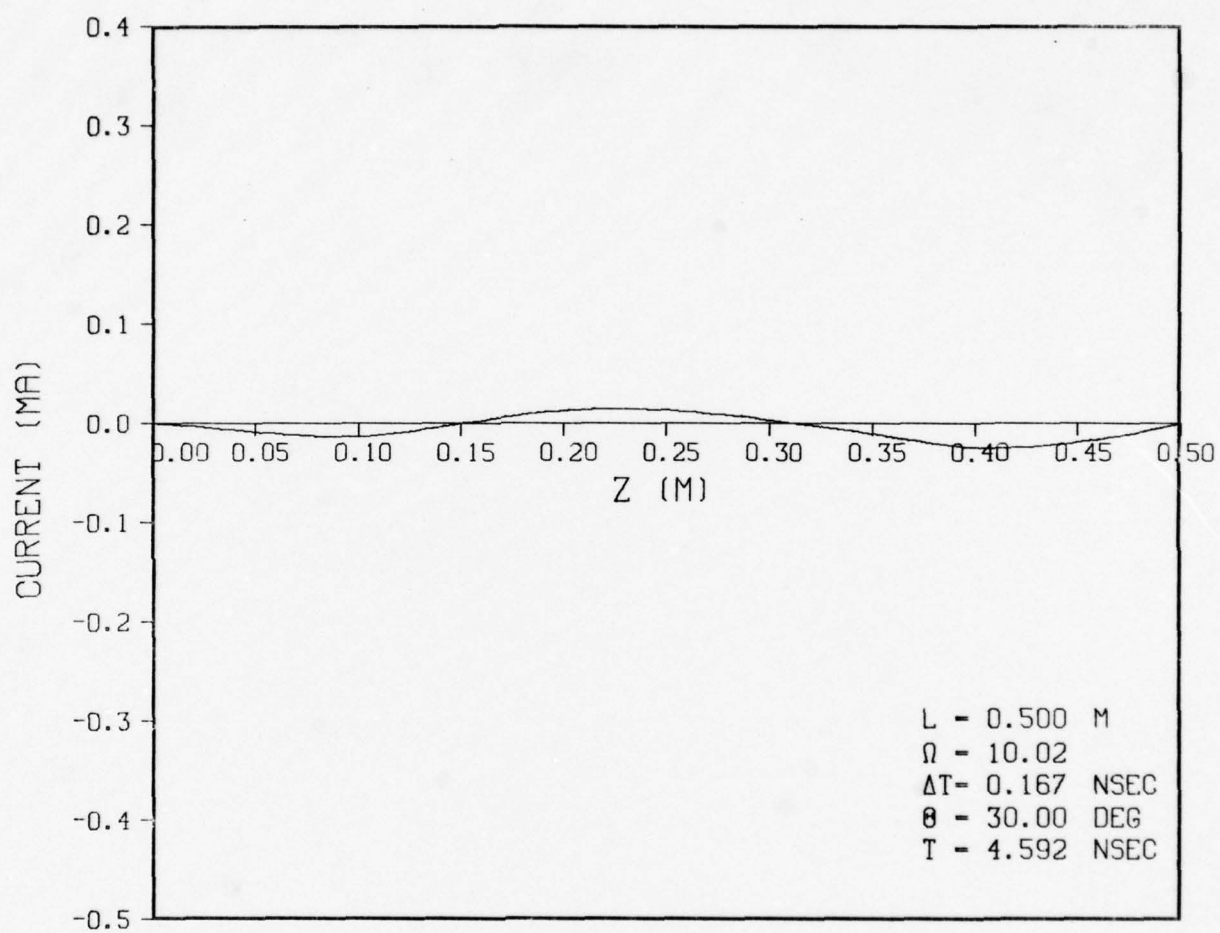


Figure 5-14. Current Distribution — Gaussian Pulse



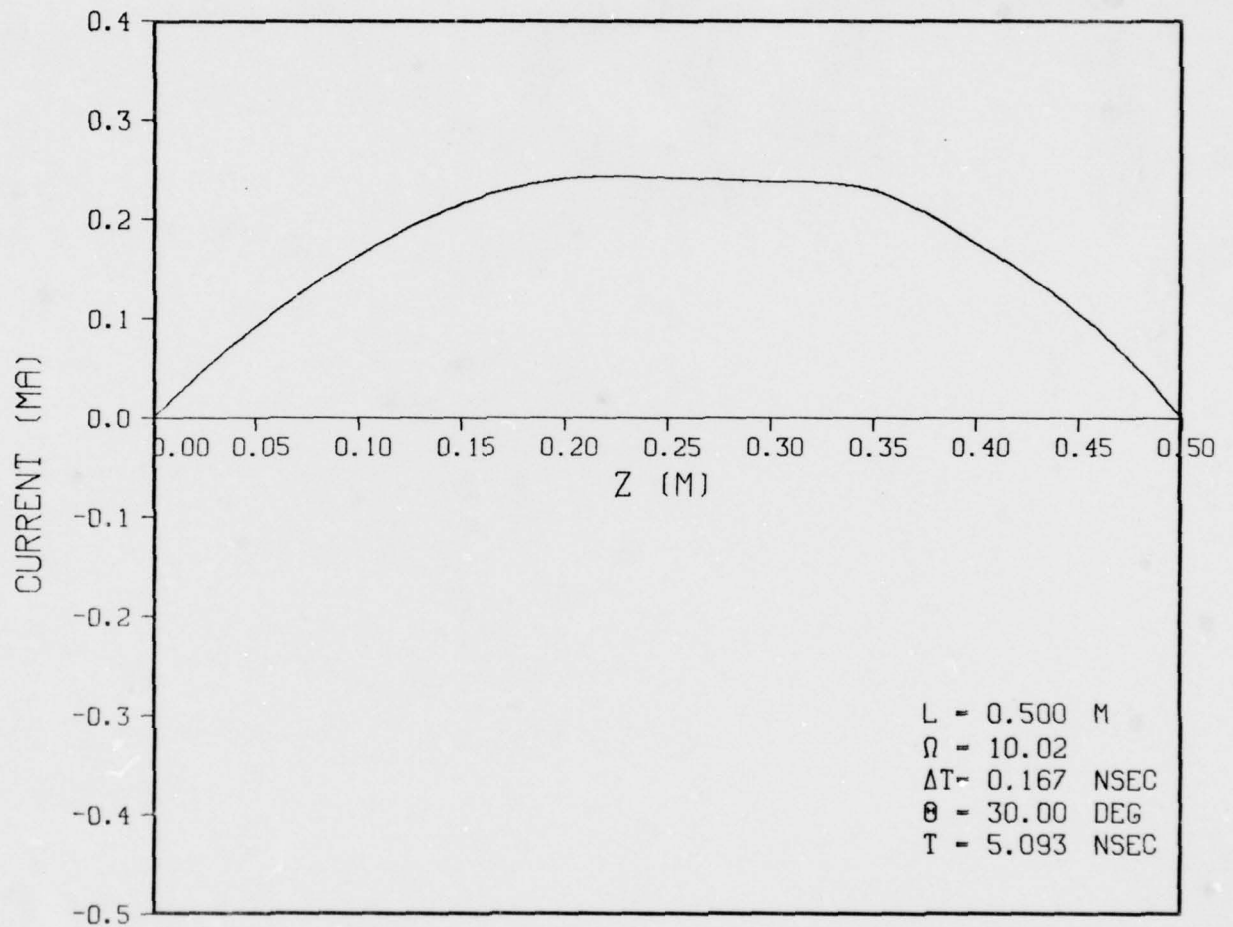


Figure 5-15. Current Distribution — Gaussian Pulse

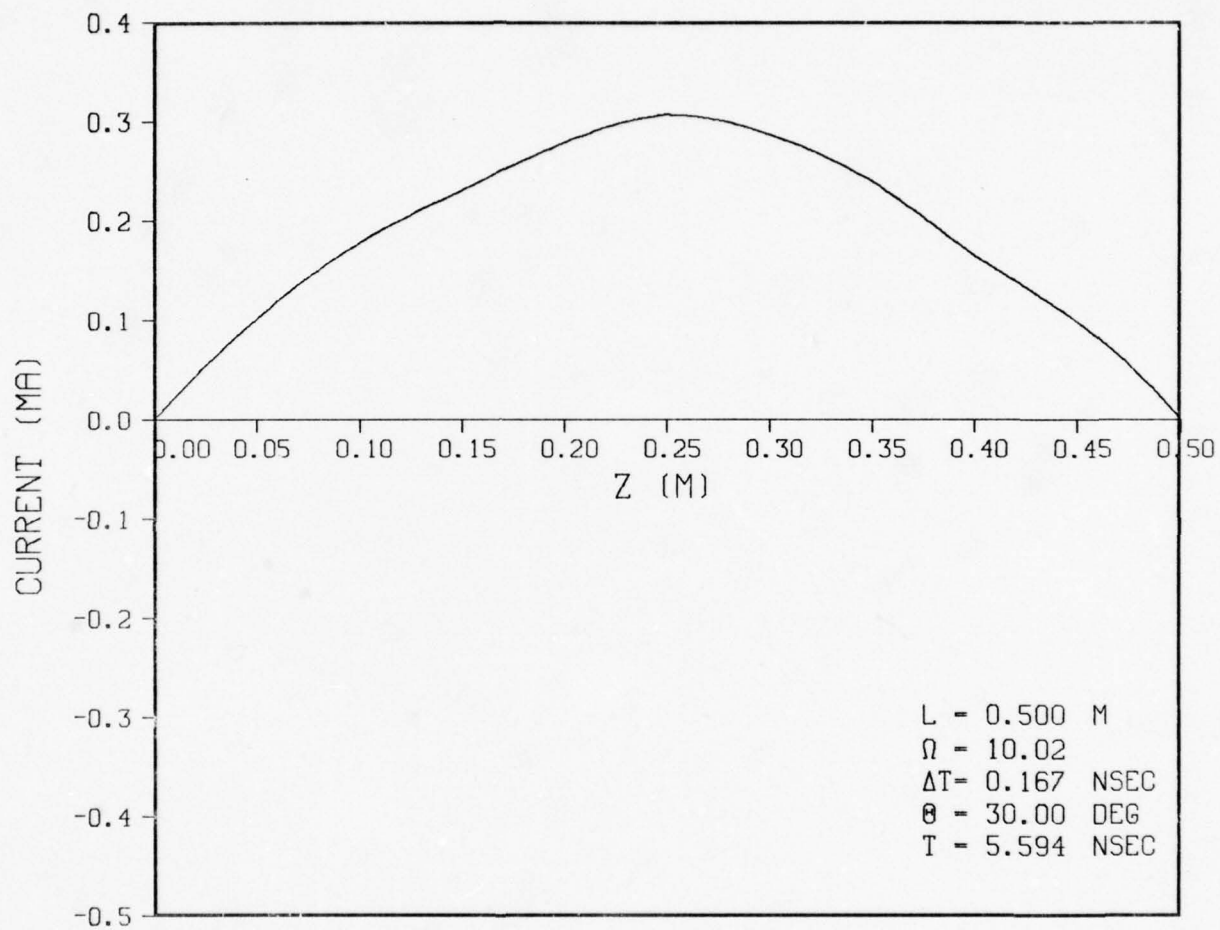


Figure 5-16. Current Distribution — Gaussian Pulse

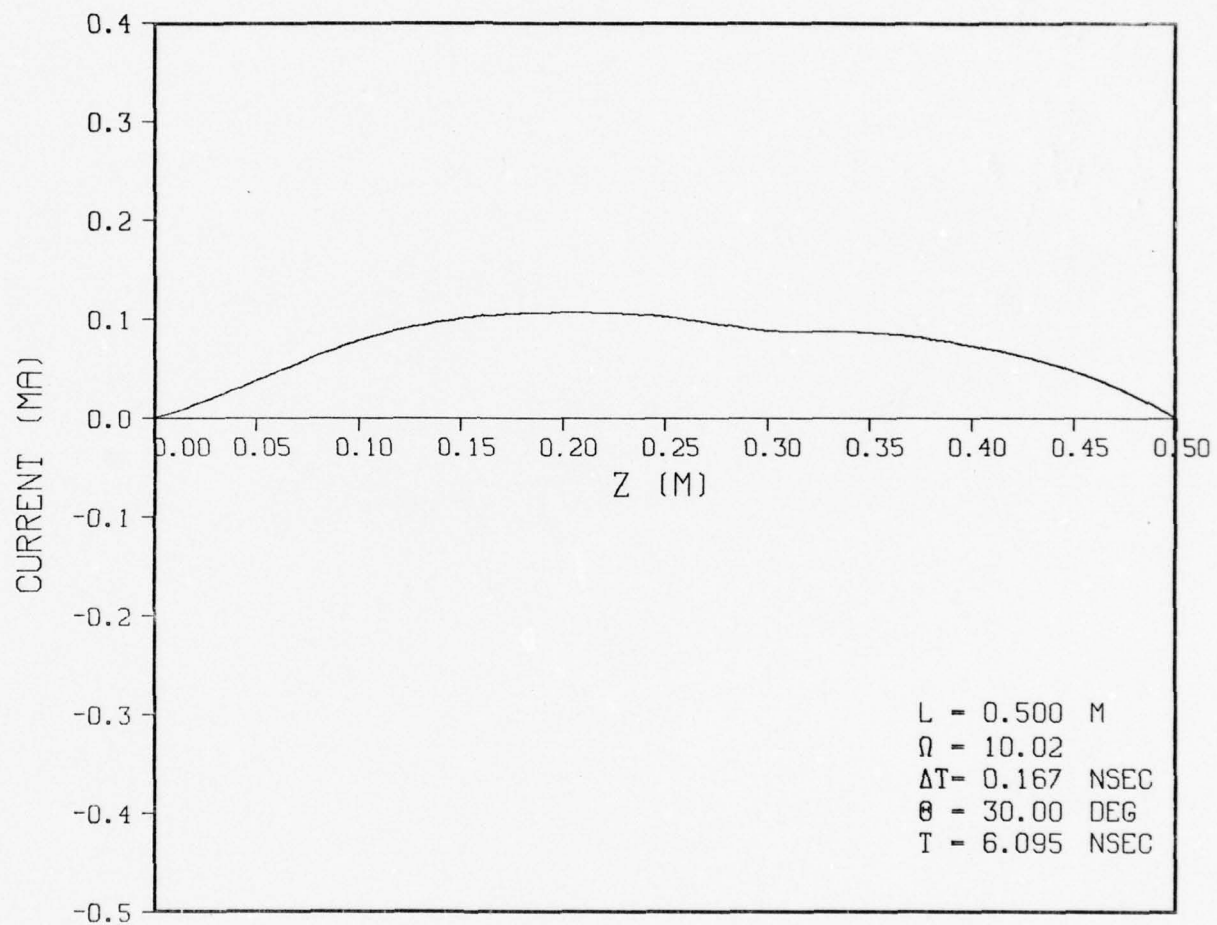


Figure 5-17. Current Distribution — Gaussian Pulse

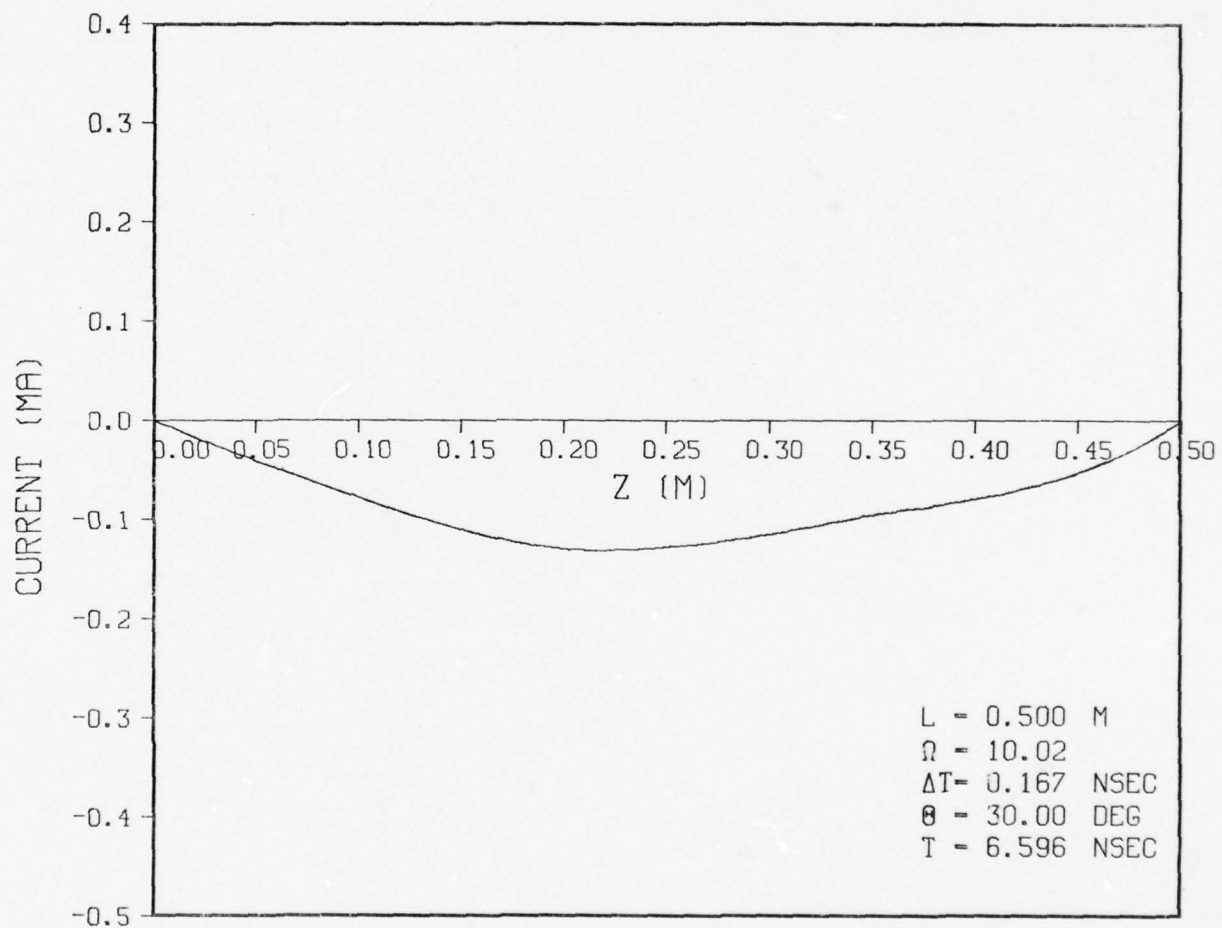


Figure 5-18. Current Distribution — Gaussian Pulse



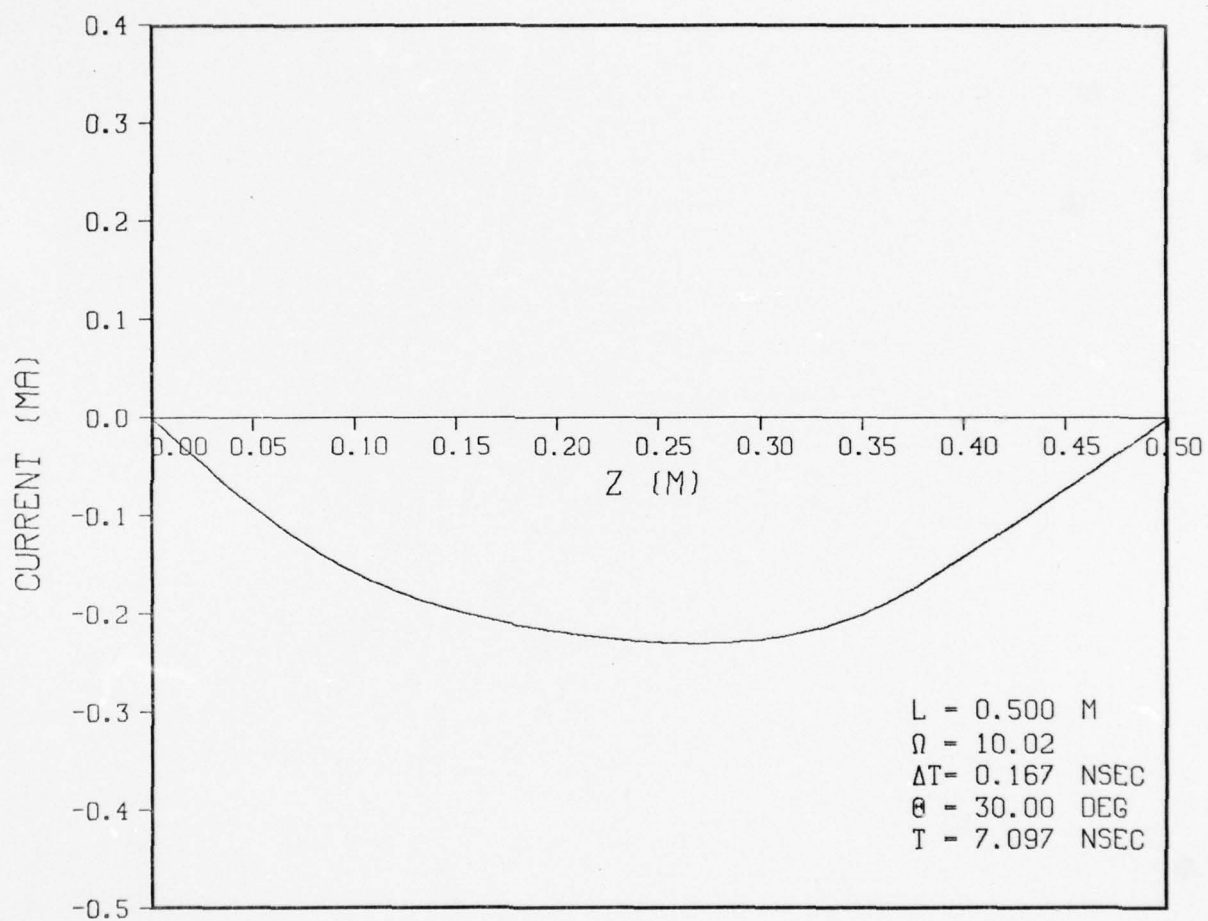


Figure 5-19. Current Distribution — Gaussian Pulse

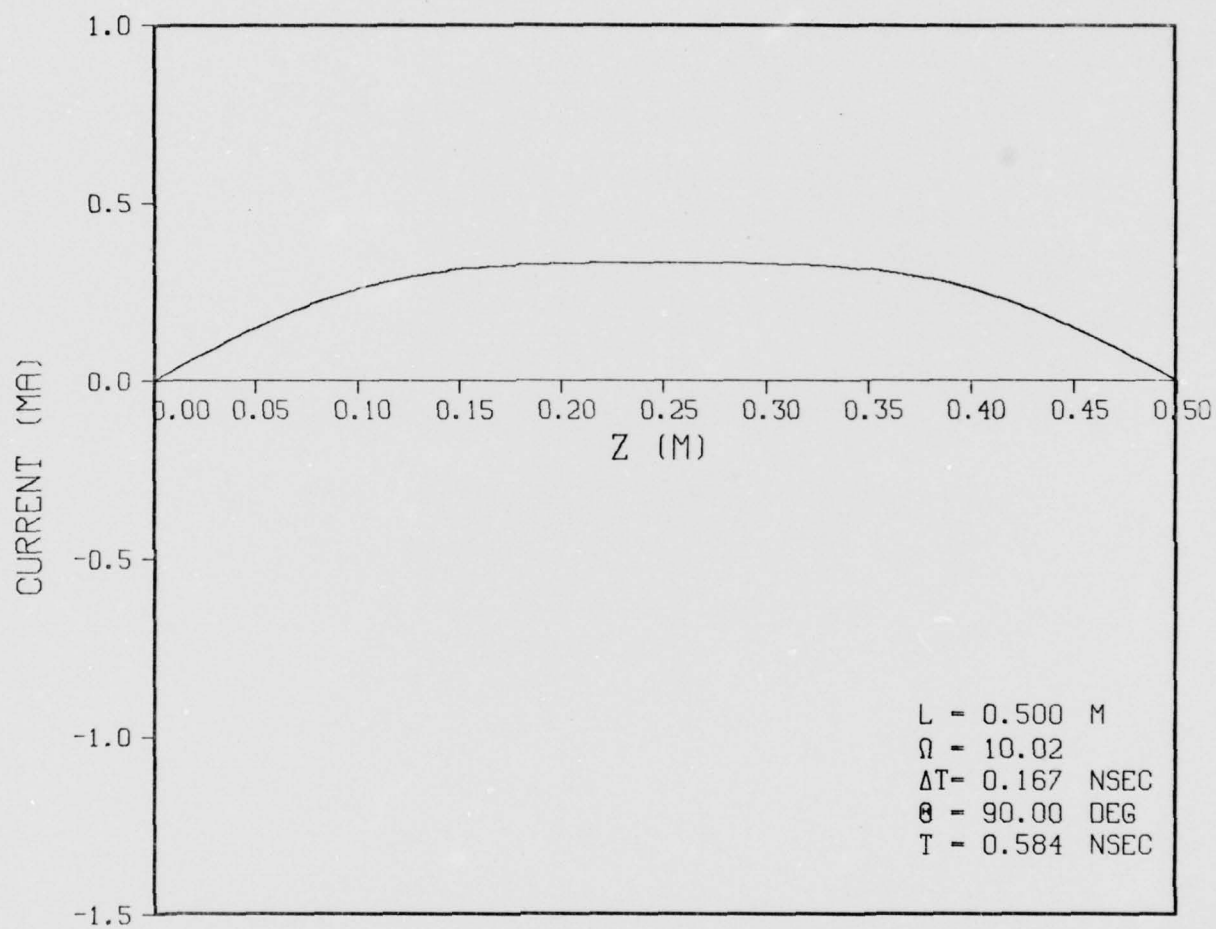


Figure 5-20. Current Distribution — Gaussian Pulse

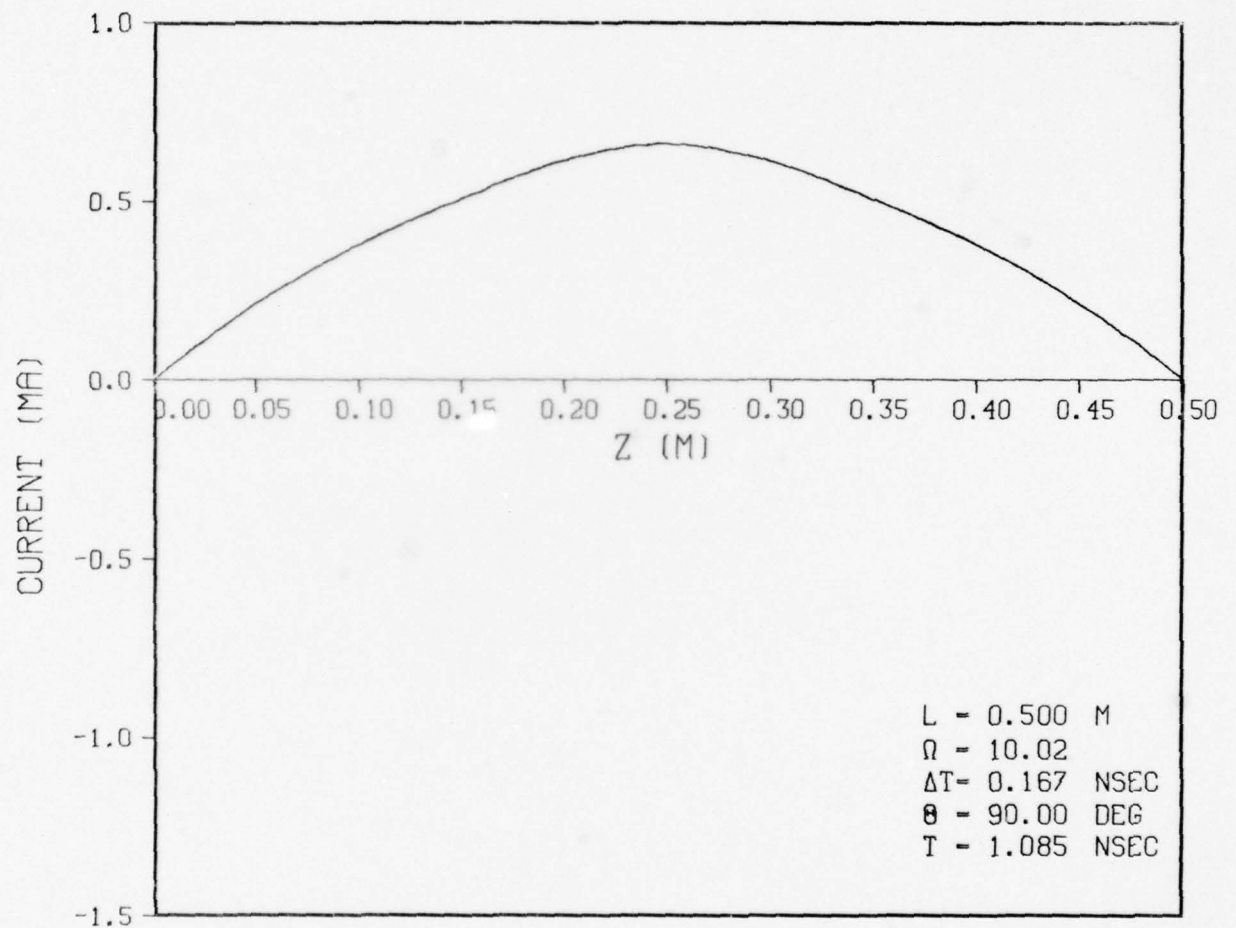


Figure 5-21. Current Distribution - Gaussian Pulse

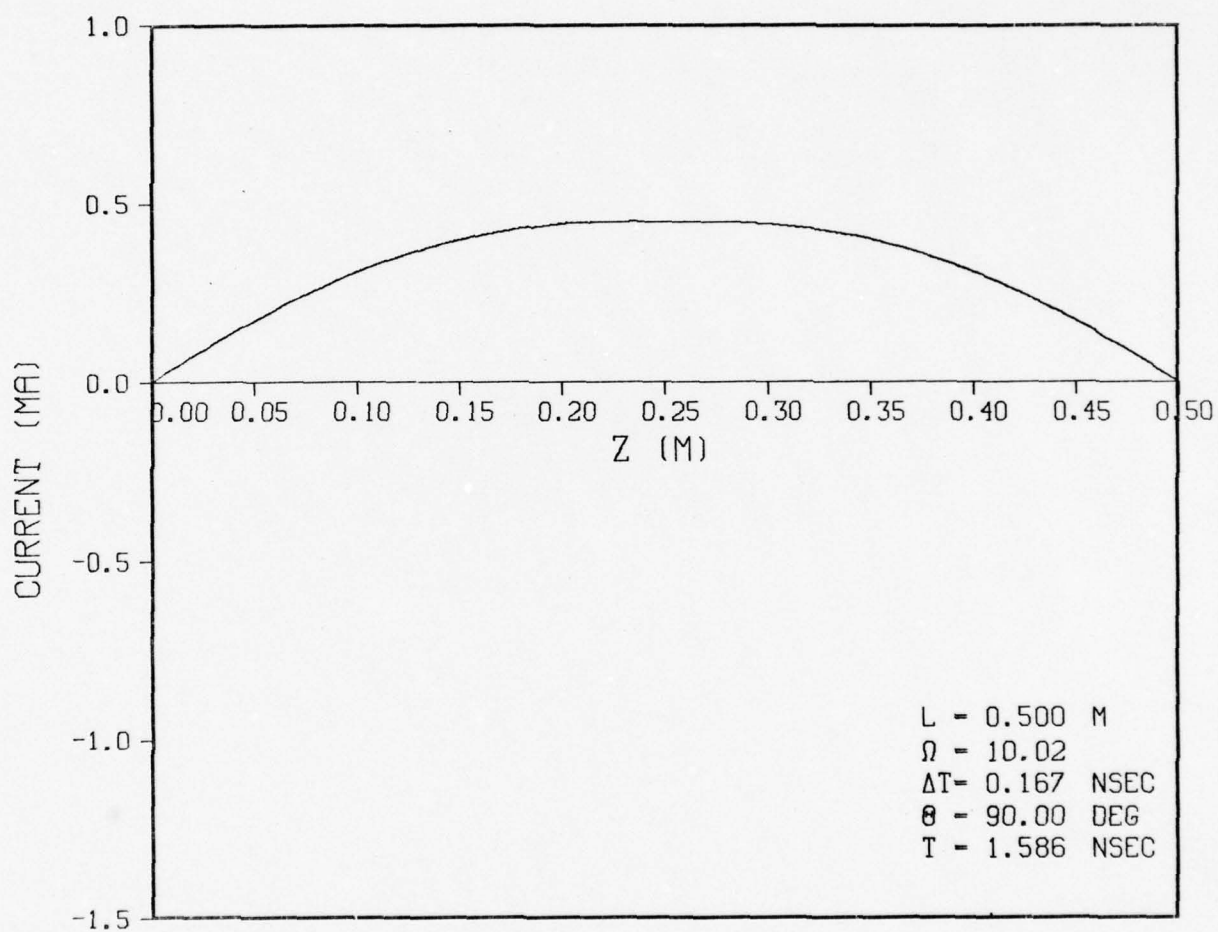


Figure 5-22. Current Distribution — Gaussian Pulse



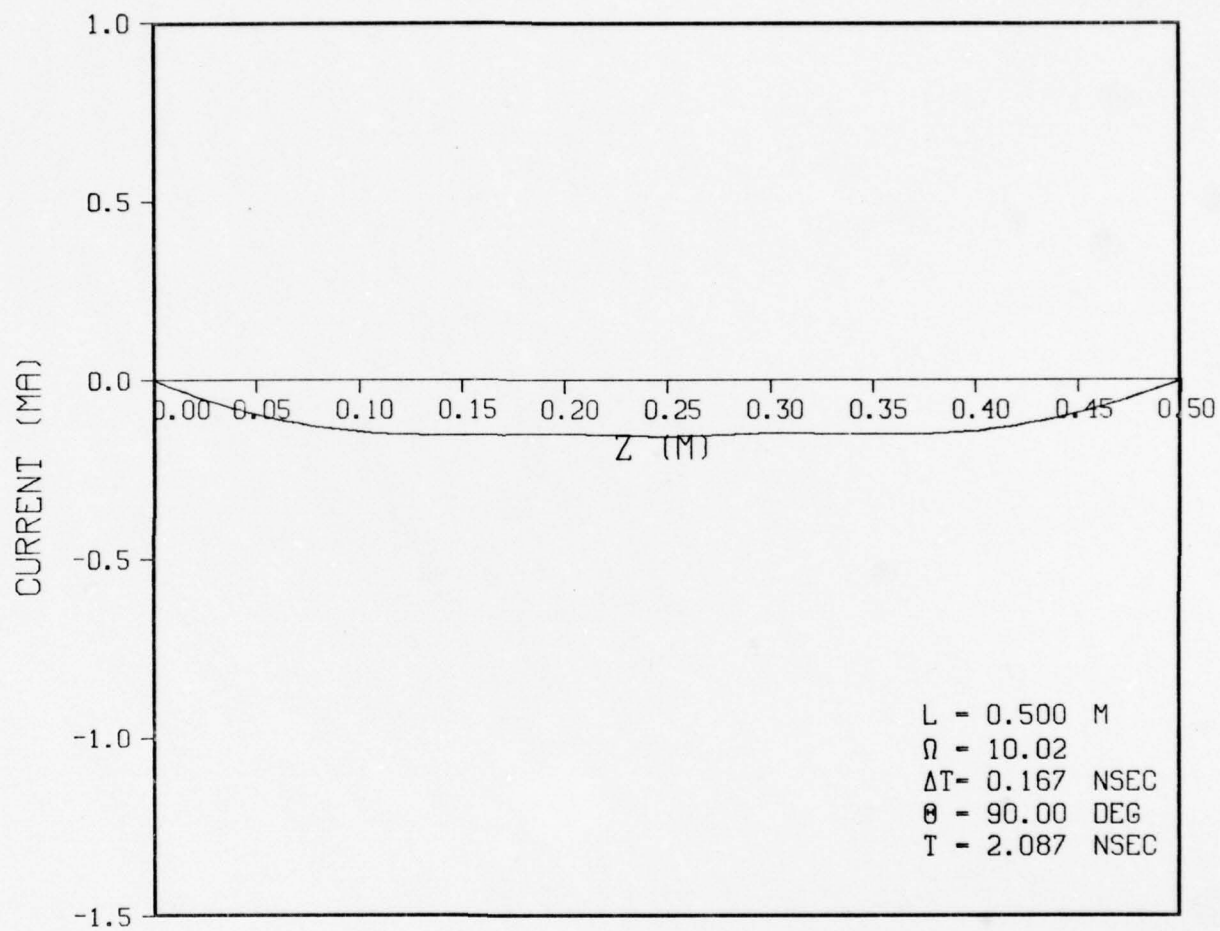


Figure 5-23. Current Distribution — Gaussian Pulse

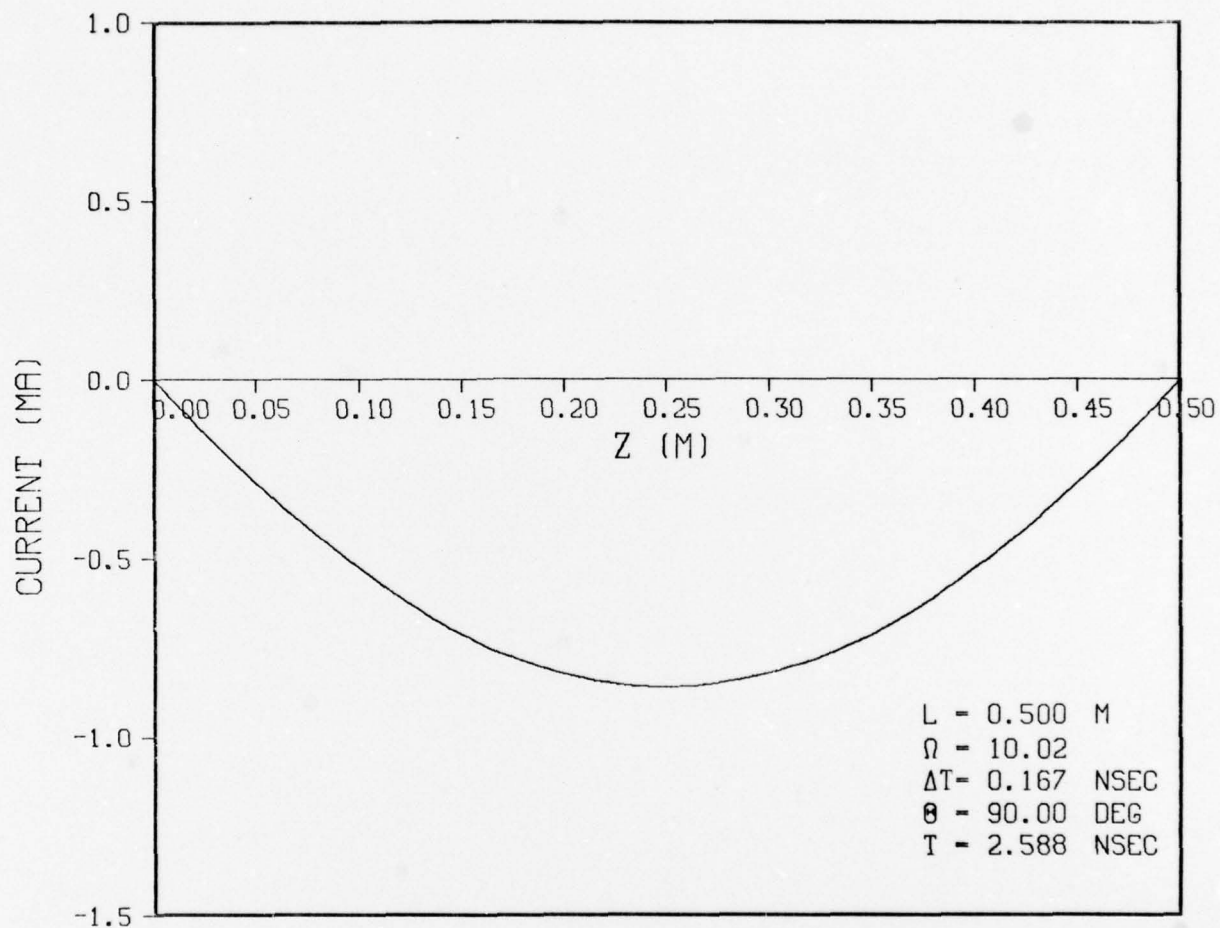


Figure 5-24. Current Distribution — Gaussian Pulse

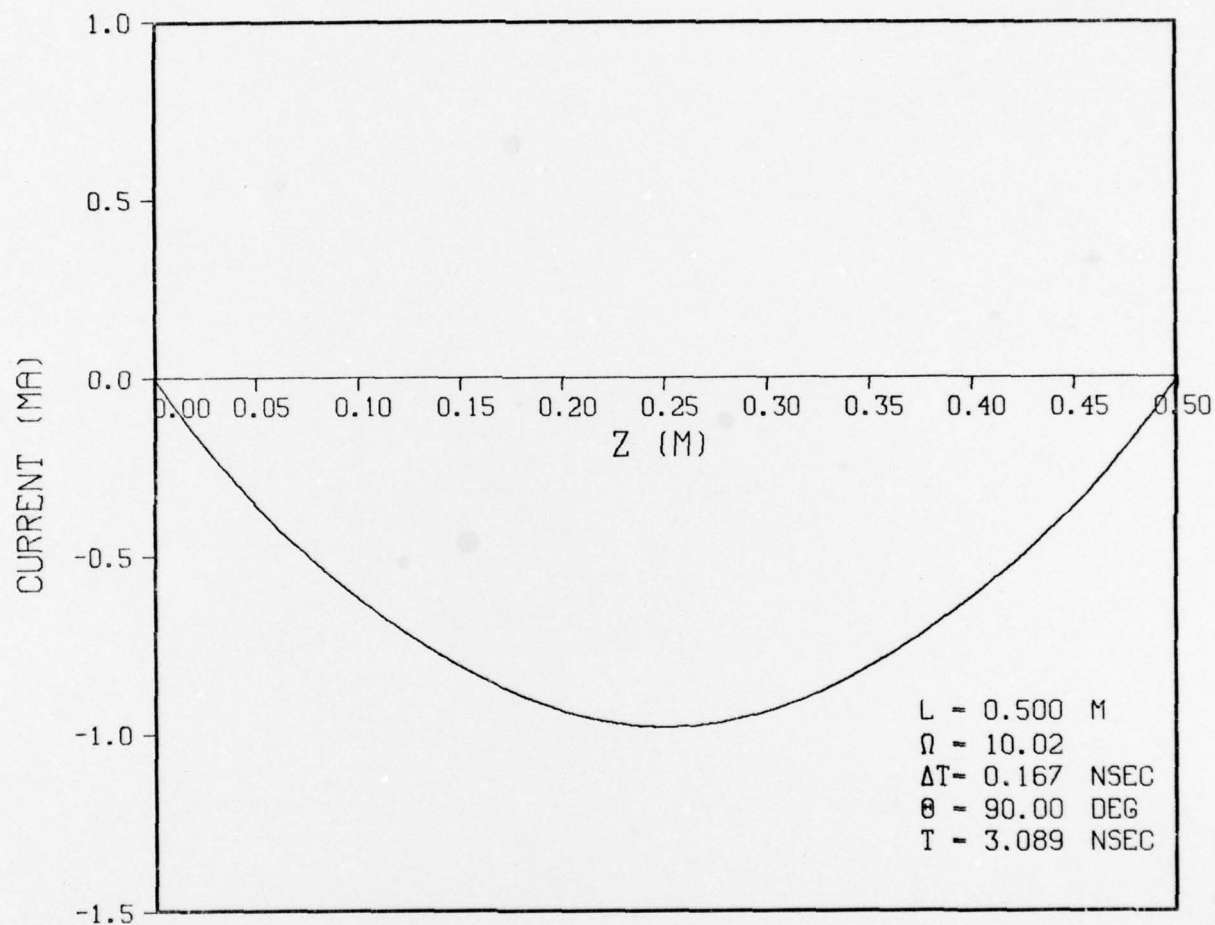


Figure 5-25. Current Distribution — Gaussian Pulse

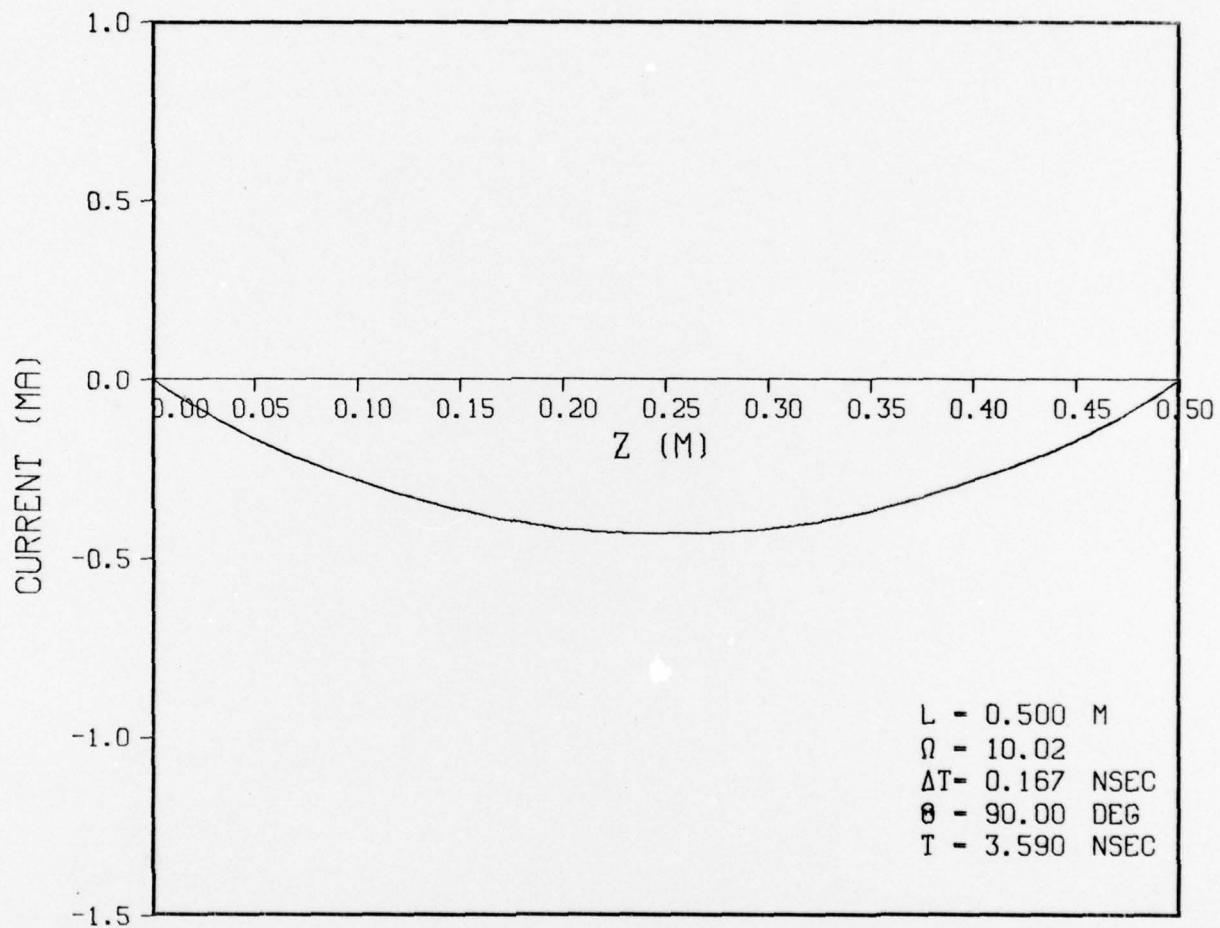


Figure 5-26. Current Distribution — Gaussian Pulse



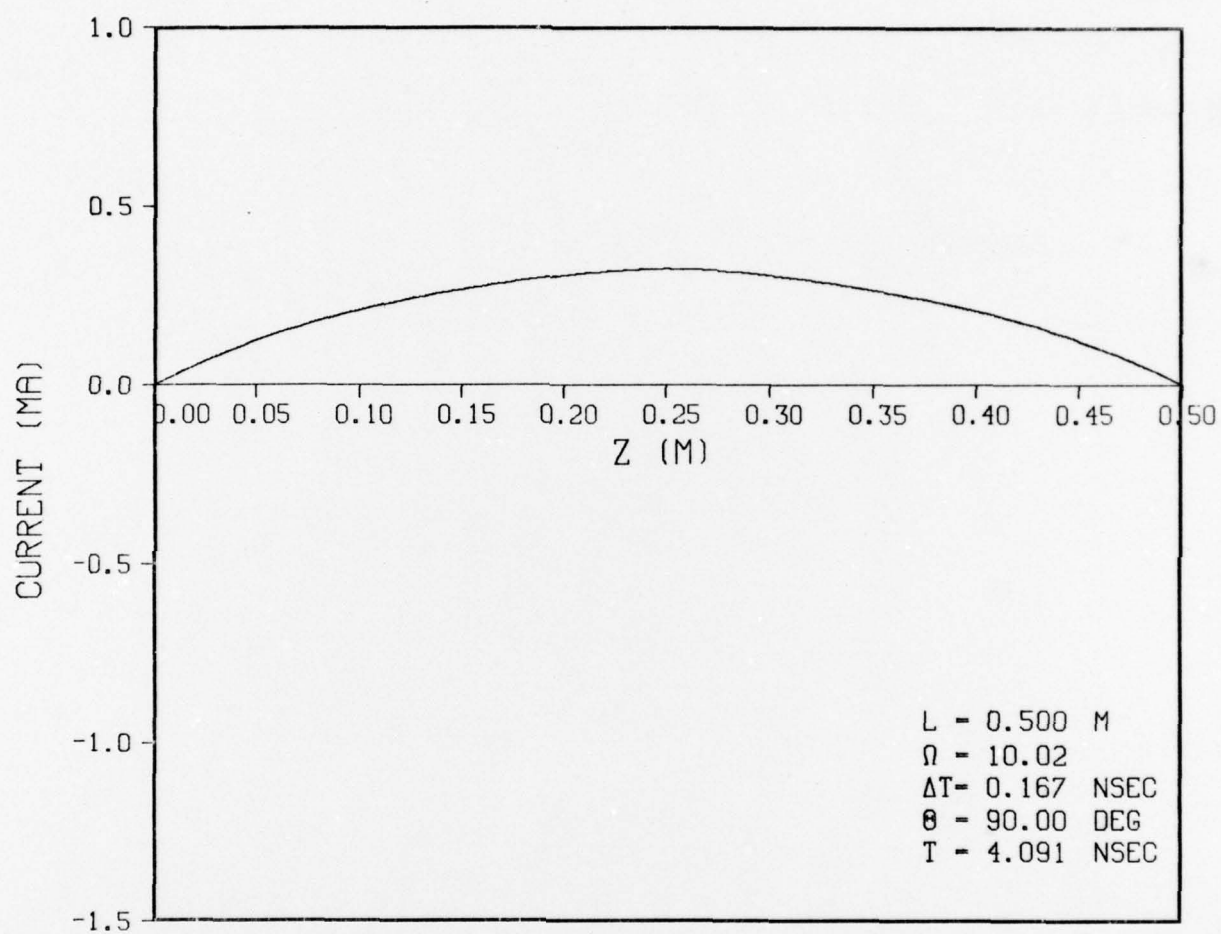


Figure 5-27. Current Distribution — Gaussian Pulse

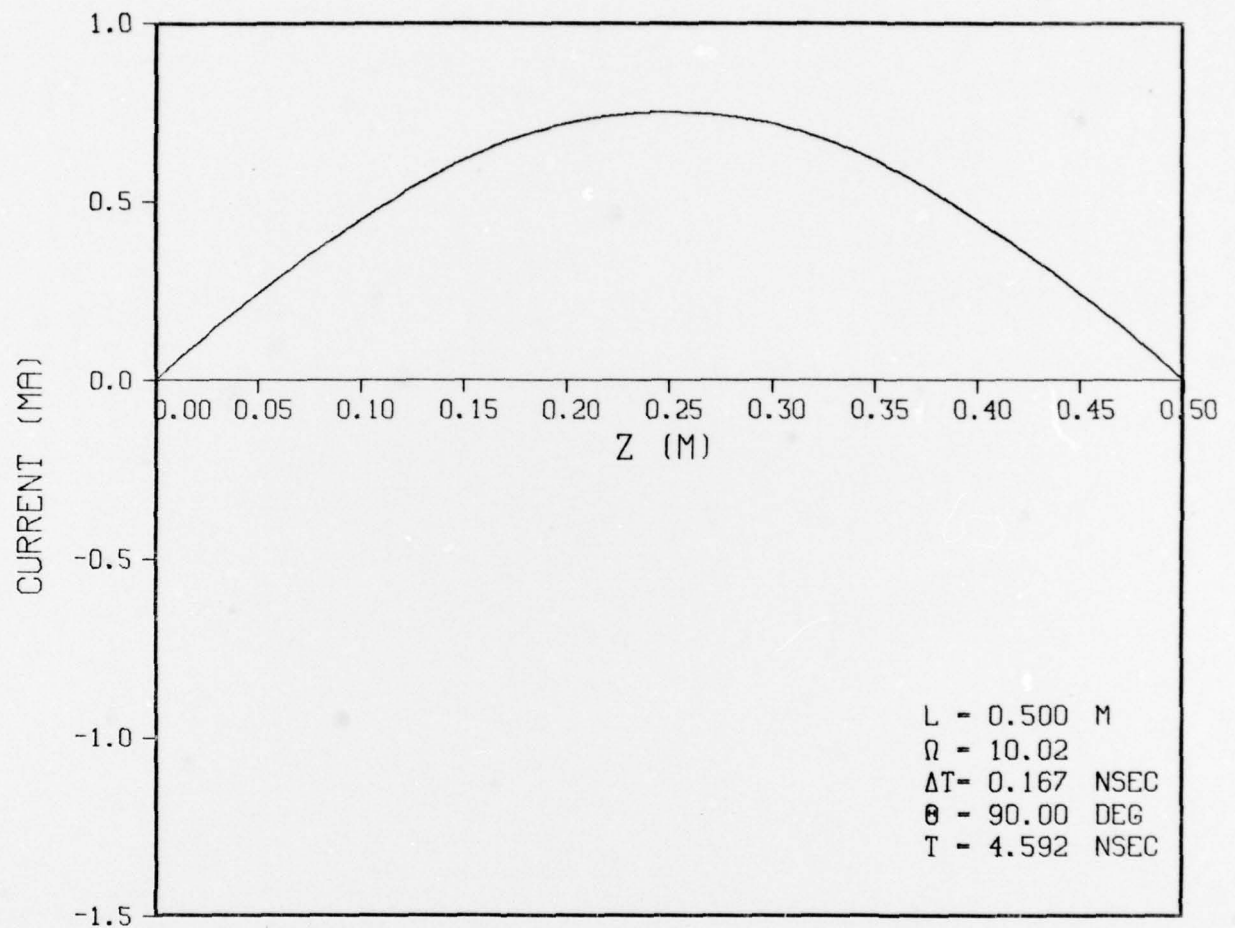


Figure 5-28. Current Distribution — Gaussian Pulse

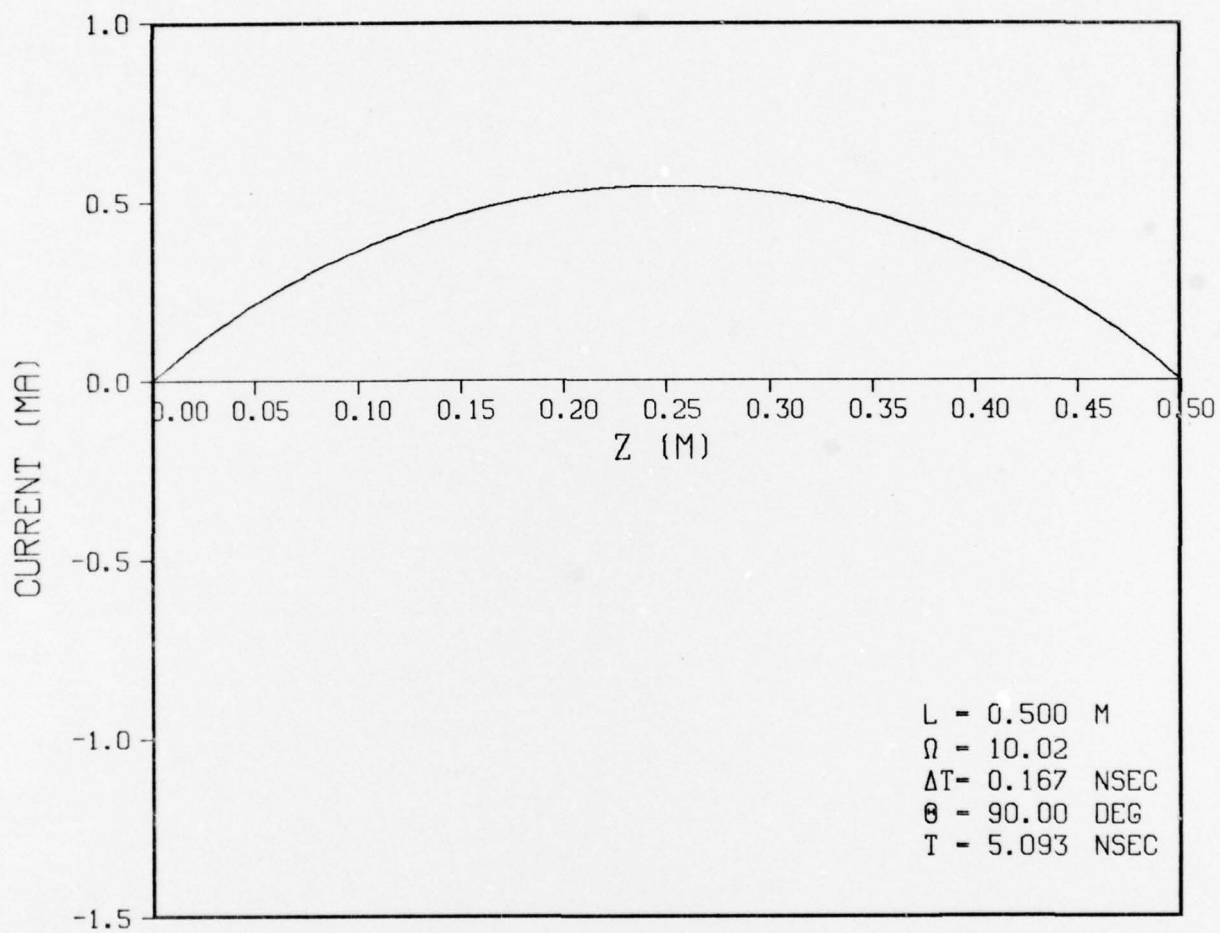


Figure 5-29. Current Distribution — Gaussian Pulse

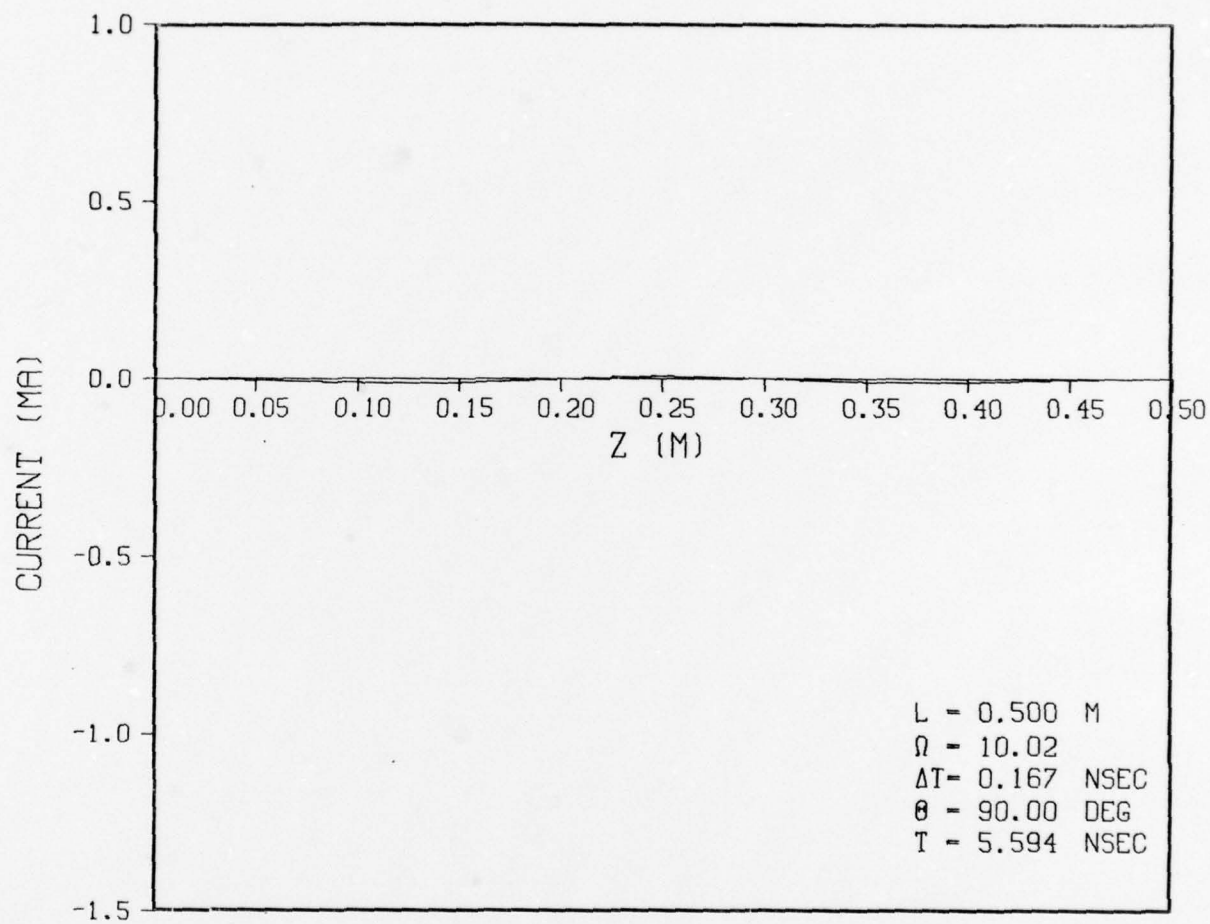


Figure 5-30. Current Distribution — Gaussian Pulse



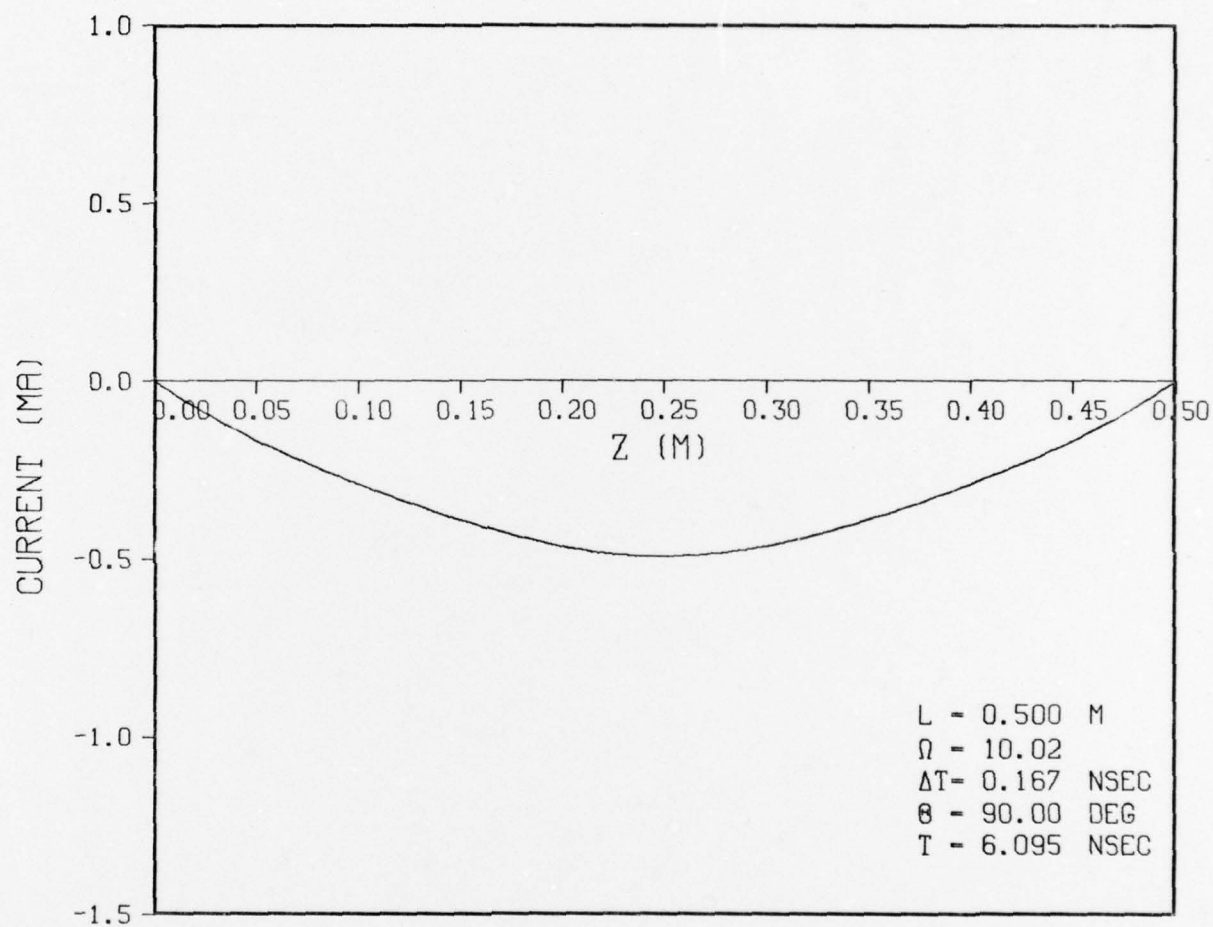


Figure 5-31. Current Distribution — Gaussian Pulse

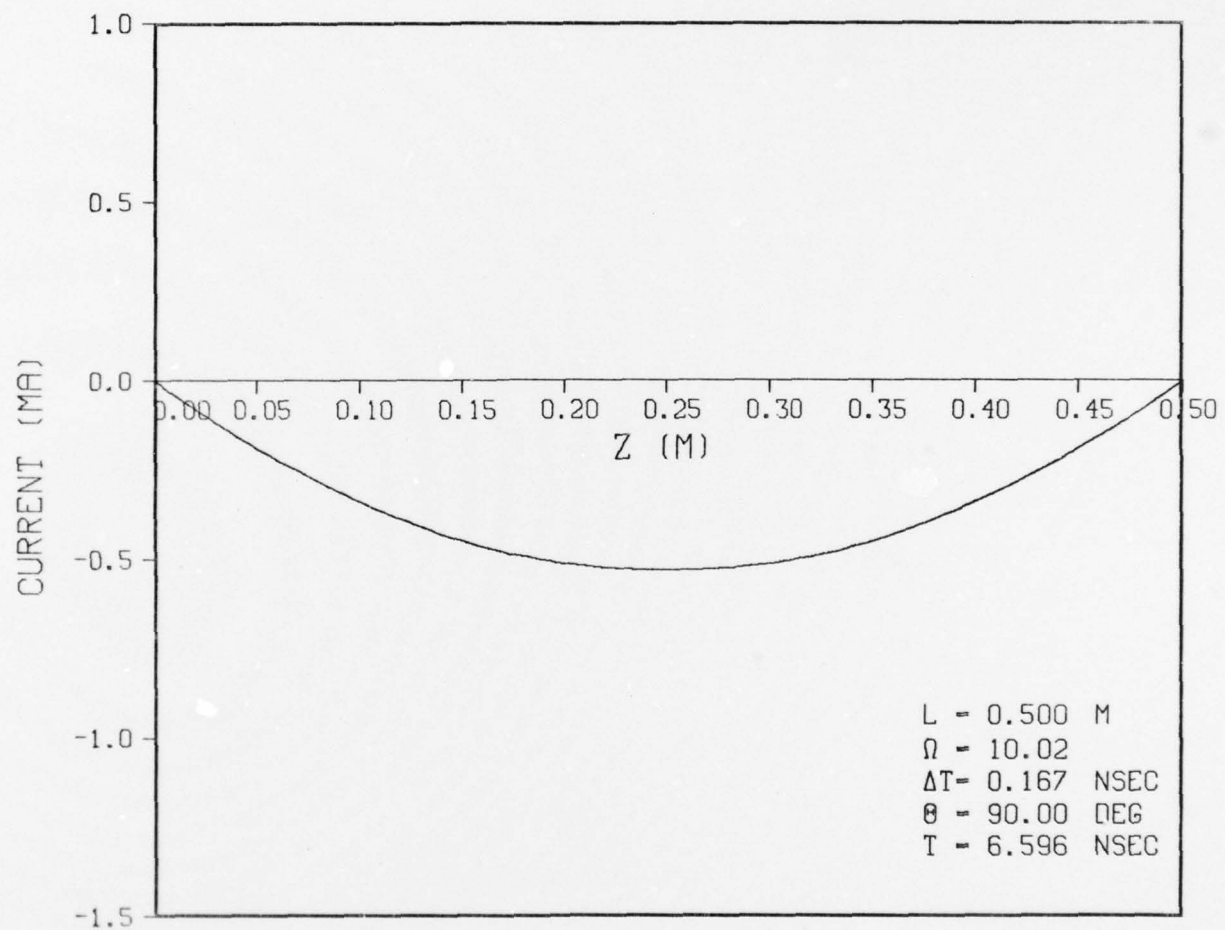


Figure 5-32. Current Distribution — Gaussian Pulse

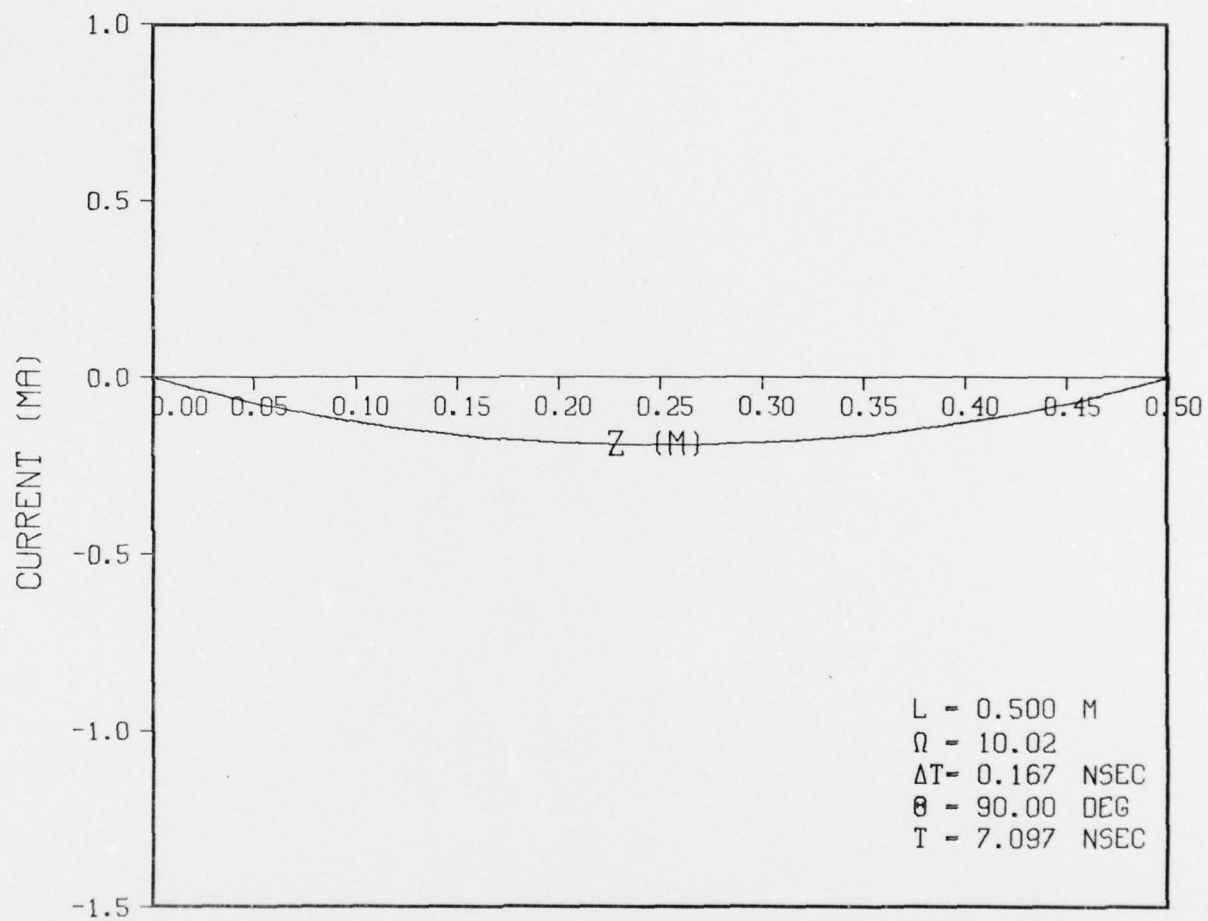


Figure 5-33. Current Distribution — Gaussian Pulse

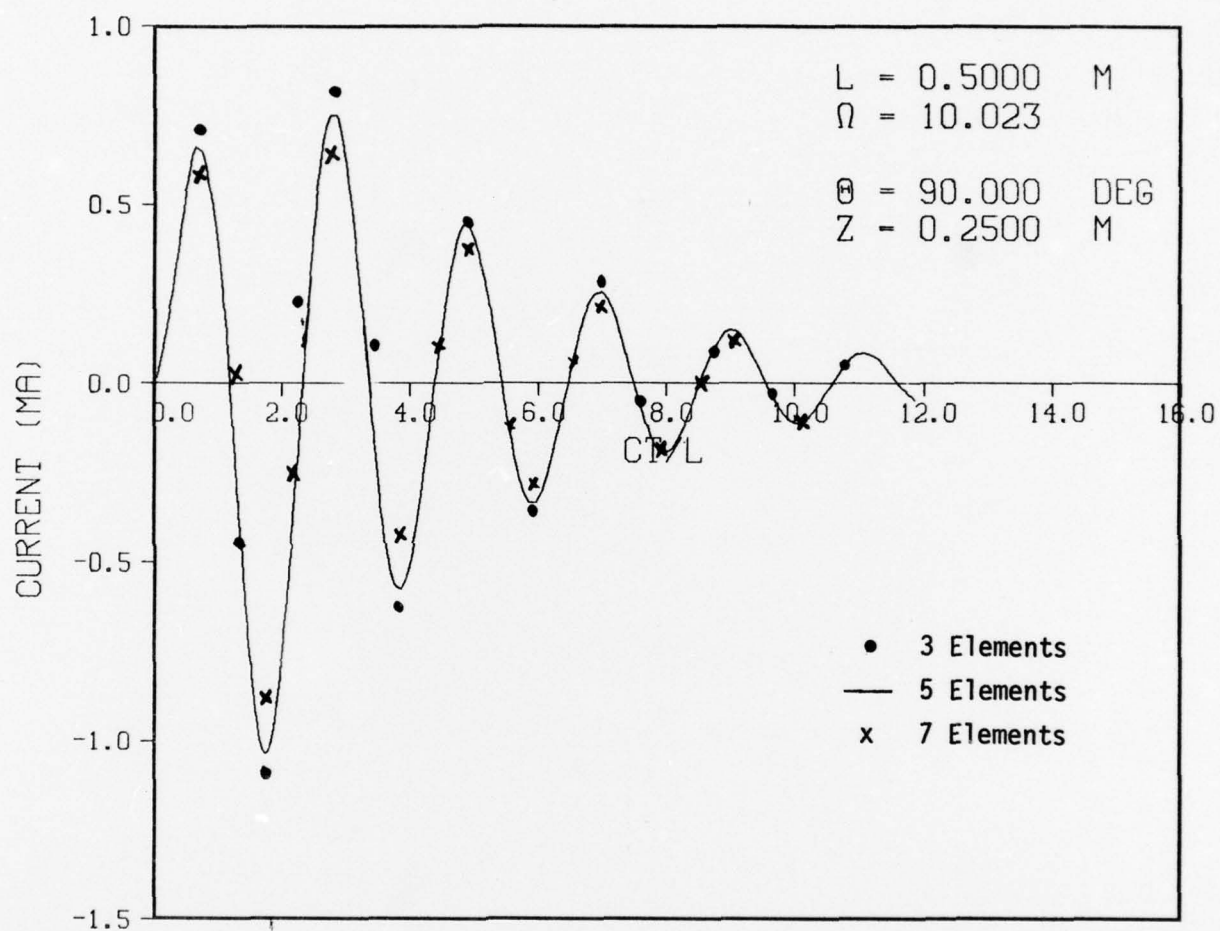


Figure 5-34. Convergence Test for the Transient Current



## 6. CONCLUSIONS AND RECOMMENDATIONS

In this study the finite element method for electromagnetic techniques has been developed to solve transient scattering problems directly in the time domain. The problem is formulated in terms of a variational time-dependent integro-differential equation which is to be solved by a finite difference scheme in time and a finite element technique in space. Based on this approach a computer program is written to calculate the transient current on thin-wire scatterers when excited by a plane wave gaussian pulse. Numerical results show good accuracy and convergence for the FEM approach. Thus, it transpires that the FEM can be a good numerical tool in solving transient electromagnetic problems. As a numerical method for solving electromagnetic scattering problems, the FEM offers the following advantages:

- (1) Since the formulation is based on the variational principle, the solution is more stable and the error is minimized.
- (2) Although the region is divided into finite elements, the whole domain remains as a continuum because of the imposed restriction on the continuity across element interfaces. This is contrary to the point-matching solution used in the method of moments where the true solution is valid only at the matching points in the whole domain.
- (3) FEM approach is particularly useful in handling complex geometries.

In spite of its advantages, the FEM-based time-domain code developed here has two shortcomings from a numerical solution point of view. The shortcomings are:

- (1) The mathematical and bookkeeping aspects of the FEM are involved, and
- (2) The computational time seems to be longer as the code is not optimized.

It is therefore hoped that further research work in this area would alleviate these difficulties.

To make this code more usable and application-oriented, it is recommended that this code (a) be extended to arbitrarily-oriented wires and other complex structures; (b) be generalized to arbitrary incident pulse; and (c) be optimized for more efficiency.

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